On the type-definability of the binding group in simple theories

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Abstract

Let T be simple, work in $\mathcal{C}^{eq}$ over a boundedly closed set. Let $p \in S(\emptyset)$ be internal in a quasi-stably-embedded type-definable set Q (e.g. Q is definable or stably-embedded) and suppose (p, Q) is ACL-embedded in Q (see definitions below). Then $Aut(p/Q) = \{\sigma|p^{\mathcal{C}} \mid \sigma \in Aut(\mathcal{C}/Q)\}$ with its action on $p^{\mathcal{C}}$ is type-definable in $\mathcal{C}^{eq}$ over $\emptyset$. In particular, if $p \in S(\emptyset)$ is internal in a stably-embedded type-definable set Q, and $p^{\mathcal{C}} \cup Q$ is stably-embedded, then $Aut(p/Q)$ is type-definable with its action on $p^{\mathcal{C}}$.

1 Introduction

Hrushovski introduced the binding group in [H1] where he proved that unidimensional stable theories are superstable. He has also used it for the general study of almost orthogonality. The central definition of internality makes sense in the simple context as well: We say that $p \in S(A)$ *is* Q*-internal* if for every realization a of p there is b with $a \underset{A}{\downarrow} b$ and a tuple c of realizations of q with $a \in \text{dcl}(b, c)$. Throughout we adopt the convention that we are working in a large, saturated model $\mathcal{C}$ of a complete first order theory with elimination of imaginaries, and we will say explicitly when using hyperimaginaries. p will always be a complete type and Q will be an invariant set, both over $\emptyset$. *The binding group $Aut(p/Q)$ of p over Q*, is the group formed by all permutations of $p^{\mathcal{C}}$ which are restrictions of A-automorphisms of $\mathcal{C}$ which fixes Q pointwise. One form of the binding group question is: Under what circumstances is the binding group and its associated action on $p^{\mathcal{C}}$ type-definable? In [H1], Hrushovski showed that in a stable theory, p (stationary) is Q-internal iff the binding group $Aut(p/Q)$ is type-definable with its action on $p^{\mathcal{C}}$. The proof relied on the definability of types and, as with many other things in the simple context, approaches to the binding group in simple theories are more difficult.

We would like to say a few words about the approach we take to defining the binding group in this paper. First, we will say that *p is controlled by a set C over Q* (and *C is a controlling set for p over Q)* if any automorphism of $\mathcal{C}$ which fixes Q and C pointwise, fixes all realizations of p. An example worth remembering in the simple context is the random bipartite graph. If p is the 1-type over the empty set in one of the partitions and q is the 1-type over the empty set in the other, then p is controlled by the empty set over $q^{\mathcal{C}}$. Nevertheless, p is foreign to q.

The approach we take is to examine circumstances under which there is a controlling set for p over Q when p is Q-internal. It is not known if this is always true. The following fact, due to the second author ([S2], Theorem 5.6) and Wagner ([W], Proposition 3.4.9), shows that it is almost true.

Fact 1.1 *Suppose $p \in S(\emptyset)$ is an amalgamation base and p is Q-internal. Then there is a Morley sequence $\bar{a}$ in p and there is a definable relation $R(x, \bar{y}, \bar{a})$ such that, for every tuple $\bar{c}$, $R(\mathcal{C}, \bar{c}, \bar{a})$ is finite and for every a' realizing p, there is some tuple $\bar{c}$ from Q such that $R(a', \bar{c}, \bar{a})$ holds. In this case we say that $R(x, \bar{y}, \bar{a})$ is a definable one-to-finite relation that covers p by Q (or that $R(x, \bar{y}, \bar{a})$ is a definable one-to-finite cover of p by Q).*

In fact, recently some stronger (and hyperimaginary) versions of this fact has been proved in [S3]. The following version [S3, Theorem 2.2] is the easiest one which will be convenient to apply.

Theorem 1.2 *Let T be simple. Let $p(x) \in S(\emptyset)$ be a hyperimaginary amalgamation base and let $\mathcal{R}$ be an invariant set of hyperimaginaries. Suppose* *that p is almost $\mathcal{R}$-internal. Then for every Morley sequence $\bar{a}$ (of length ω) of p there is a type-definable one-to-bounded relation $S(x, \bar{y}, \bar{a})$ (i.e. for every $\bar{y}$ there are boundedly many x-s for which $S(x, \bar{y}, \bar{a})$ holds) which covers p by $\mathcal{R}$. If p and $\mathcal{R}$ are real then S can be chosen to be definable.*

As a consequence, with p, Q and I as in the above Fact, the automorphism group of p over Q fixing I is contained in a product of $\mathcal{C}$-many finite groups. It is worth remarking that in the stable case, if p is Q-internal then p is controlled over Q and moreover, *p is generated over Q* i.e. $p(\mathcal{C}) \subseteq dcl(Q \cup A)$ for some set A. In the simple context, Pillay has an example in which p is Q-internal but is not generated over Q. However, it remains an open question whether if p is Q-internal, p is controlled over Q in any simple theory. The results presented here suggest a quite general assumption in which internality implies controlled, generalizing results from [S2] (e.g. stable-embeddedness of Q and $p^{\mathcal{C}} \cup Q$ is a special case of our assumption).

2 Controlling Sets

In this section, we show how the techniques used for the multiplicity one case and in the case of internality via a foreign generic parameter, both found in [S2], can be extended. Throughout this section T is assumed to be simple, $p \in S(\emptyset)$ an amalgamation base and Q an arbitrary $\emptyset$-invariant set. We assume that all sets are sets of hyperimaginaries and that all types are types of hyperimaginaries. First, a reminder of some definitions. Suppose p is Q-internal. We say that *$p_b \in S(b)$ is a witness for the internality of p in Q* if p_b doesn't fork over $\emptyset$ and there is $a \models p_b$ and tuple c from Q such that $a \in \mathrm{dcl}(b, c)$.

Here is the condition that will be operative for this section.

Definition 2.1 Suppose p is Q-internal. We say that *p is Q-internal via a disjoint generic parameter modulo p* if for some I, a Morley sequence in p and some witness p_b for the internality of p in Q , with $b \perp\!\!\!\perp I$, $tp(\sigma(b)/Ib) \perp^a p_b$ for all automorphisms σ which fix I and Q

The following fact from [S2] plays an important role in the upcoming lemma.

Fact 2.2 *Suppose* $p_b \in S(b)$ *is a witness for the internality of* p *in* Q*. Fix* $\sigma \in Aut(\mathcal{C}/Q)$*. Then for every* $a \models p_b$ *the type* $tp(a/b, \sigma(b))$ *determines the type* $tp(a, \sigma(a)/b, \sigma(b))$ *i.e. if* a' *is such that* a *and* a' *have the same type over* $b, \sigma(b)$ *then* $(a, \sigma(a))$ *and* $(a', \sigma(a'))$ *have the same type over* $b, \sigma(b)$*.*

For an infinite indiscernible sequence I of realizations of p we denote by p_I the limit type of I, that is, the complete type over I of any element a such that $I^\frown a$ is indiscernible.

Lemma 2.3 *Assume that* p *is* Q*-internal via a disjoint generic parameter modulo* p *with the notation as given in Definition 2.1. Let* $B = (b_i | i < |T|^+)$ *be an independent set of realizations of* $tp(b)$*. Then* $B \cup I$ *controls* p_I *over* Q*.*

Proof: Let $\sigma \in Aut(\mathcal{C}/Q \cup I \cup B)$ and let $a \models p_{_I}$. By the independence theorem we may assume $a \models p_{b'}$ (where $p_{b'}$ is the corresponding conjugate of p_b) and $a \mathop{\smile\!\!\!\!\!\;|}\; Ib'$. Suppose $\sigma(a) \neq a$. Let $i^* < |T|^+$ be such that b_{i^*} is independent from $I \cup \{b', \sigma(b')\}$ over $\emptyset$. Now, $p_{b_{i^*}}$, $p_{_I}$ have a common non-forking extension say $\bar{p}_{_I}$. By the assumption $\tilde{p}_{_I} \equiv tp(a/I, b', \sigma(b'))$ doesn't fork over $\emptyset$. By the independence theorem $\tilde{p}_{_I}$ and $\bar{p}_{_I}$ have a common non-forking extension. Let a' realize this extension. Then by Fact 1.3, on one hand $a' \models \tilde{p}_{_I}$ implies $\sigma(a') \neq a'$ where on the other hand $a' \models \bar{p}_{_I}$ implies $\sigma(a') = a'$, a contradiction. Hence $\sigma(a) = a$. □

Theorem 2.4 *Assume* p *is* Q*-internal via a disjoint generic parameter modulo* p *with the notation as given in Definition 2.1. Let* $B = (b_i | i < |T|^+)$ *be an independent set of realizations of* $tp(b)$*. Then* $B \cup I$ *controls* $p^{\mathcal{C}}$ *over* Q*.*

Notation: For any (not necessarily small) set X, we define $DCL(X)$ as the set of all elements fixed by all automorphisms of $\mathcal{C}$ which fix X pointwise.

Proof: By Lemma 2.3, $p_{_I}^{\mathcal{C}} \subseteq DCL(Q \cup B \cup I)$. Hence, by applying Lemma 2.3 again, we conclude that for every sequence J of realizations of p_I which is a Morley sequence conjugate to I, $p_{_J}^{\mathcal{C}} \subseteq DCL(Q \cup B \cup I)$. By applying Lemma 2.3 for the third time we conclude that for every sequence K of realizations of p_J which is a Morley sequence conjugate to I, $p_{_K}^{\mathcal{C}} \subseteq DCL(Q \cup B \cup I)$. Now, let a be an arbitrary realization of p. Then there exists J as above with the additional property that J is independent from a over $\emptyset$, denote this sequence by J^*. Now, observe that since p is an amalgamation base (i.e. all realizations of p have the same Lascar strong type over $\emptyset$), there exists K', a conjugate of I such that $K'^\wedge a$ is an indiscernible sequence and $tp(K'/a)$ realize over $\emptyset$ every desired $Lstp(I')$ for conjugate I' of I (indeed, if I' is an arbitrary given conjugate of I, choose a' such that $I'^\wedge a'$ is indiscernible sequence. By the fundamental properties of $Lstp$ there exists K' such that $Lstp(K'^\wedge a) = Lstp(I'^\wedge a')$). Thus we may apply the independence theorem to find K^* such that both $J^{*\wedge} K^*$ and $K^{*\wedge} a$ are indiscernible. $\square$

Remark 2.5 We say that p is Q-internal via a disjoint generic parameter (with respect to conjugation over Q) if there is a witness $p_b \in S(b)$ for the internality of p in Q such that p_b is almost orthogonal to $\sum_Q(y, b)$ over b, i.e. $a \underset{b}{\downarrow} b'$ for every $a \models p_b$ and $b' \models \sum_Q(y, b)$ (where $\sum_Q$ is the invariant relation defined by $\sum_Q(b, b')$ iff b, b' are Q-conjugate). We say that p has (internal) multiplicity m in Q, if m is the minimal natural number such that there is Morley sequence $(a_i|i \leq \omega)$ of p and tuple c from Q such that $mult(a_\omega/(a_i|i < \omega) \cup c) = m$ (such m always exists). Note that internality of (internal) multiplicity 1 is a special case of internality via a disjoint generic parameter modulo p. In [S2] it was shown that in each of the above cases p is controlled over Q.

We give here an example of an application of Theorem 2.4. In the following, $bdd(Q) = \{d|d \in bdd(\bar{c})$ for some tuple $\bar{c}$ from $Q\}$.

Claim 2.6 *Suppose $p \in S(\emptyset)$ is an amalgamation base which is Q-internal, $p^{\mathcal{C}} \not\subseteq bdd(Q)$, and this internality is witnessed by $p_b \in S(b)$. Then for every Morley sequence I of p and for every realization b' of $tp(b)$ with $b' \downarrow I$ and $\sigma \in Aut(\mathcal{C}/Q^{\mathcal{C}} \cup I)$, $\sigma(b') \underset{I}{\not\downarrow} b'$.*

Proof: By Theorem 1.2, for every Morley sequence I of p there is a type-definable one-to-bounded cover $R(x, z, I)$ of p by Q. Let b' be a realization of $tp(b)$ independent from I over $\emptyset$. Let $p_{b'}$ be the corresponding conjugate (over $\emptyset$) of p_b. By the independence theorem, there is a common nonforking extension of the limit type p_I of I and $p_{b'}$. Let a realize this common extension. Let $\bar{c}$ be a tuple from Q such that $a = f(\bar{c}, b')$ for some type-definable partial function $f(z, y)$ (over $\emptyset$) and such that $a \in R(\mathcal{C}, \bar{c}, I)$. Since $a \downarrow b'I$ and we are assuming $p \not\subseteq bdd(Q)$, $\bar{c} \underset{Ia}{\not\downarrow} b'$. Let $F = R(\mathcal{C}, \bar{c}, I) \cap p_I^{\mathcal{C}}$. Let $\langle b_i'|i < \lambda\rangle$ be a sufficiently long Morley sequence of any nonforking extension

over $I \cup F$ of $tp(b'/I, a)$. We may assume that $b'_0 = b'$ (by conjugating $\bar{c}$.) Let $\sum_{Q,I}(y, y')$ be the invariant equivalence relation (over I) expressing that y, y' are conjugate over $Q^{\mathcal{C}} \cup I$.
Subclaim The b'_i-s are pairwise non-$\sum_{Q,I}$-equivalent. First, note:

Remark: Let $E(x, y)$ be an invariant equivalence relation. Let a be (of the same sort as x) such that a/E has unboundedly many conjugates. Then if $E(a, b)$ holds then $a \not\downarrow b$.

By the Subclaim and this remark, for every $\sigma \in Aut(\mathcal{C}/Q^{\mathcal{C}} \cup I)$, $\sigma(b') \underset{I}{\not\downarrow} b'$.

Proof of Subclaim Suppose that the b'_i-s are $\sum_{Q,I}$-equivalent. Hence for every i, $f(\bar{c}, b'_i) \in R(\mathcal{C}, \bar{c}, I)$. Let $S \subseteq \lambda$ be of size λ such that for every $i \in S$, $f(\bar{c}, b'_i) = a'$ and $tp(a', b'_i/\bar{c}, I) = tp(a, b'_0/\bar{c}, I)$ for some fixed $a' \in F$. Hence $\bar{c} \underset{Ia'}{\not\downarrow} b'_{s_0}$, where $s_0 = \min(S)$. A contradiction to the fact that $\langle b'_i | i \in S\rangle$ is a Morley sequence of $tp(b'_{s_0}/I, a')$. $\square$

Example 2.7 Suppose $p \in S(\emptyset)$ is Q-internal via a p-regular generic parameter (i.e. there exists a witness p_b for the internality of p in Q such that if $q = tp(b)$ then every forking extension of q is orthogonal to p). Moreover assume $p^{\mathcal{C}} \not\subseteq bdd(Q)$. Then p is controlled over Q.

Proof: Follows immediately by Theorem 2.4 and Claim 2.6.

3 The assumption of ACL-embeddedness of (p, Q) in Q

In this section we work in $\mathcal{C}^{eq}$ over a boundedly closed set (i.e. every hyperimaginary that is in the bounded closure of the set is already in the definable closure of it) in a saturated model of a simple theory. Thus, every set, type etc. will be of imaginaries unless stated otherwise. Here $p \in S(\emptyset)$ and Q a type-definable set over $\emptyset$. The following notation is convenient.

Notation: For any (not necessarily small) set X, we define:

1. $ACL(X)$ is the set of all hyperimaginaries with finite orbits under automorphisms that fix X pointwise.

2. $BDD(X)$ is the set of all hyperimaginaries with bounded orbits under automorphisms that fix X pointwise.

3. $DCL^{veq}(X)$ is the set of all objects of the form a/E that are fixed by every automorphism that fixes X pointwise, where a is some (possibly infinite) tuple of imaginary elements and E is a directed union of $\emptyset$-definable equivalence relations. Likewise we define $ACL^{veq}(X)$.

Definition 3.1 *Suppose p is Q-internal. We say that (p, Q) is ACL-embedded in Q if there is a Morley sequence I of p such that $DCL^{veq}(p^{\mathcal{C}} \cup Q) \subseteq ACL^{veq}(Q \cup I)$.*

Remark 3.2 1) Note that if the internal multiplicity (for definition see Remark 2.5) of $p \in S(\emptyset)$ in Q is 1, then (p, Q) is ACL-embedded in Q. 2) In section 4 (Theorems 4.1, 4.4), we suggest two general special cases where (p, Q) is ACL-embedded in Q.

Proof: 1) If the internal multiplicity of p in Q is 1, then there a definable function f defined over some initial segment of a Morley sequence I of p such that $f[Q^n] \supseteq p^{\mathcal{C}}$ (for some $n < \omega$). In particular,

$$DCL^{veq}(p^{\mathcal{C}} \cup Q) \subseteq DCL^{veq}(dcl(Q \cup I)) = DCL^{veq}(Q \cup I).$$

□

Here is an example due to Shelah of a simple theory with (p, Q), where p is Q-internal but (p, Q) is not ACL-embedded in Q.

Example 3.3 Consider the complete bipartite graph with partitions I and J and the generic q-colouring $\nu : I \times J \to GF(q)$ (where $q = p^n$ for some prime p). Let V be a vector space over $GF(q)$ freely generated by the symbols $\{v_j : j \in J\}$. We interpret I as a set of functionals on V, and V as a set of functions from I to $GF(q)$ by: for $i \in I$ and $v = \sum_{j \in J} k_j v_j \in V$ (where $k_j \in GF(q)$) let

$$\langle i, v \rangle = \sum_{j \in J} k_j \nu(i, j).$$

Let $M = (I, V, GF(q), \langle , \rangle)$ (the language L has 3 sorts and contains a constant symbol for each $a \in GF(q)$). By this interpretation we can define in L linear independence over $GF(q)$ in V (note that every set of distinct elements in I is linearly independent).

Claim 3.4 *For every $n < \omega$, for every linearly independent $v_0, ..., v_n \in V$ over $GF(q)$ and for every $a_0, ...a_n \in GF(q)$, there exists $i^* \in I$ such that $\langle i^*, v_j \rangle = a_j$ for every $j \leq n$.*

Proof: Easy linear algebra.

Note that by Claim 3.4, $Th(M)$ is $\aleph_0$-categorical, has elimination of quantifiers and can be axiomatized by saying that V contains (infinitely many) realizations of each finite conjunction of formulas of the form $\langle x, i \rangle = a_i$ (for distinct $i \in I$ and $a_i \in GF(q)$), the content of the above claim, and the sentences saying that the elements of V (viewed as functions from I to $GF(q)$) are closed under linear combinations. Now, using elimination of quantifiers and Claim 3.4, it is not hard to see that M has D-rank 1.

The counterexample to ACL-embeddedness is interpretable in the previous structure. Let $P = I \times \mathrm{GF}(q)$, $Q = I$ and $R = V$. Only the affine structure on R will be present, there will be a projection π from P to Q ($\pi(i, l) = i$), and the only other relation is a function $f : Q \times R \to P$ defined by

$$f(q, r) = (q, \langle q, r \rangle).$$

The structure we consider is therefore $N = (P, Q, R, \pi, f)$. It is reasonably easy to see that P is Q-internal.

Now $P = \pi^{-1}(Q)$ and each fiber $\pi^{-1}(q)$ has q elements. If P is fixed by an automorphism then so is all of R so R is contained in $DCL(P \cup Q)$. But if we simply fix Q and any Morley sequence I in p then R is not necessarily fixed. To see this, notice that for any $v \in V$ the map μ_v such that $x \mapsto x + v$ on R lifts to an elementary map which fixes Q and which (for every q) fixes $\pi^{-1}(q)$ iff $\langle q, v \rangle = 0$. If $\hat{I} = \pi(I)$ then for any $v \in V$ such that $\langle i, v \rangle = 0$ for all $i \in \hat{I}$, μ_v fixes Q and I. Hence (P, Q) is not ACL-embedded in Q.

Similarly, it also follows that $Aut(P/Q) \cong V$ and that the multiplicity of P in Q is q (Note that since (P, Q) is not ACL-embedded in Q, Remark 3.2 implies immediately that this multiplicity is > 1.)

Now, let us recall some terminology from [S3]. An $\emptyset$-invariant set Q is said to be *pseudo-stably-embedded* if for every complete type $q \in S(\emptyset)$ the equivalence relation Σ^q_Q defined on $q^{\mathcal{C}}$ by $\Sigma^q_Q(b, b')$ iff " b, b' are conjugate over Q ", is type-definable. We say that Q is *pseudo-open* if the equivalence relation E^{tp}_Q defined by " $E^{tp}_Q(b, b')$ iff $tp(b/Q) = tp(b'/Q)$ " is type-definable (globally, not restricted to a complete type). We will say that Q is *quasi-stably-embedded* if Q is pseudo-stably-embedded and each Σ_Q^q is an intersection of definable equivalence relations. Note that every definable set is pseudo-open and every stably-embedded type-definable set is quasi-stably-embedded. Moreover, by the following fact [S2, Proposition 2.5] we conclude that every definable set is quasi-stably-embedded.

Fact 3.5 *Let Q be pseudo-open $\emptyset$-invariant set and let X, X' be tuples of variables of the same sort. Then there exists a type-definable equivalence relation $\sum_Q(X, X')$ such that for all A, A', $(A, A') \models \sum_Q(X, X')$ iff there exists $\sigma \in Aut(\mathcal{C}/Q)$ such that $\sigma A = A'$ (in this case we write $TP(A/Q) = TP(A'/Q)$). If Q is open then $\sum_Q(X, X')$ is an intersection of $\emptyset$-definable equivalence relations.*

In this section we prove the following.

Theorem 3.6 *Assume Q is quasi-stably-embedded type-definable set, and p is Q-internal. Moreover, suppose (p, Q) is ACL-embedded in Q. Then*
1) There is a set (of imaginaries) $B \subseteq DCL(p^{\mathcal{C}} \cup Q)$ that controls p over Q.
2) $Aut(p/Q)$ is type-definable with its action on $p^{\mathcal{C}}$.

For proving the Theorem we need two lemmas.

Lemma 3.7 *Suppose $E(x, x')$ is a co-type-definable relation (i.e., defined by the complement of a partial type) which is a bounded equivalence relation on a partial type $\sum(x)$. Then there is a definable relation $E^*(x, x')$ which equivalent to $E(x, x')$ on $\sum(x)$ and has finitely many classes on $\sum(x)$.*

Proof: Similar to [S1], Lemma 7.

Lemma 3.8 *Assume Q is pseudo-stably-embedded, p is Q-internal, and (p, Q) is ACL-embedded in Q. Then there is a witness $p_{\tilde{b}} \in S(\tilde{b})$ for this internality, where $\tilde{b}$ is a hyperimaginary in $DCL(p^{\mathcal{C}} \cup Q)$. If Q is assumed to be quasi-stably-embedded, then $\tilde{b}$ can be chosen to be a (possibly infinite) sequence of imaginary elements.*

Proof: By our assumption, $DCL^{veq}(p^{\mathcal{C}} \cup Q) \subseteq ACL^{veq}(Q \cup I)$ for some Morley sequence I of p. Let p_b be any witness for the internality of p in Q. Let $\psi(x, \bar{z}, y)$ be a formula which describes a partial function $(\bar{z}, y) \mapsto x$

and such that for some realization of a of p_b, and a tuple $\bar{c}$ of realizations of Q, $\psi(a,\bar{c},b)$. Let us define $E(y,y')$ by: $E(b,b')$ iff for every $a'\models p$ and every tuple $\bar{c}'$ of realizations of Q, $\psi(a',\bar{c}',b)\leftrightarrow\psi(a',\bar{c}',b')$. Observe that by compactness E is a directed union of $\emptyset$-definable equivalence relations. Let $\tilde{p}\in S(I,b)$ be any non forking extension of p_b. Then $\tilde{p}$ is a witness for the internality of p in Q. Let $\sum_{Q,I}(y,y')$ be the type-definable equivalence relation over I such that for b,b' we have $\sum_{Q,I}(b,b')$ iff b, b' are conjugate over $Q\cup I$. Let $\Lambda[(y,Y),(y',Y')]$ be defined by: $Y=Y'$ and $\sum_{Q,Y}(y,y')$ and $tp(y,Y)=tp(b,I)$ and $E(y,y')$. Let $\tilde{b}=b^\wedge I/\Lambda$.

Claim 3.9 *Λ is a type-definable equivalence relation on $tp(b^\wedge I)$, $\tilde{p}|_{\tilde{b}}$ is a witness for the internality of p in Q and $\tilde{b}\in DCL(p^{\mathcal{C}}\cup Q)$. If Q is quasi-stably-embedded, then Λ is an intersection of definable equivalence relations.*

Proof: First, clearly Λ is an equivalence relation (since E is). Clearly $b/E\in DCL^{veq}(p^{\mathcal{C}}\cup Q)$. Hence by our assumption $b/E\in ACL^{veq}(Q\cup I)$, that is, $\{\sigma(b/E)|\sigma\in Aut(\mathcal{C}/Q\cup I)\}$ is finite, hence the restriction of E to $\sum_{Q,I}(y,b)$ is a finite equivalence relation. Since E is clearly co-type-definable, Lemma 3.7 implies that E is relatively definable inside $\sum_{Q,I}(y,b)$. Thus Λ is a type-definable equivalence relation on $tp(b^\wedge I)$. If we are assuming Q is quasi-stably-embedded, Λ will be an intersection of definable equivalence relations. Clearly $\tilde{b}\in DCL(p^{\mathcal{C}}\cup Q)$. It remains to check that $\tilde{p}|_{\tilde{b}}$ is a witness for the internality of p in Q. Indeed, we may assume $a\models\tilde{p}$. Then $a\perp\tilde{b}$ and it is enough to show that $a\in dcl(\tilde{b},\bar{c})$. To see that, let $\sigma\in Aut(\mathcal{C}/\tilde{b},\bar{c})$. Then $\sigma(I)=I$, $tp(\sigma b/I)=tp(b/I)$ and $E(\sigma b,b)$ and $\sum_{Q,I}(b,\sigma b)$. Since $\psi(a,\bar{c},b)$, we get $\psi(\sigma a,\bar{c},\sigma b)$. So, $E(b,\sigma b)$ implies $\sigma(a)=a$. This ends the proof of the claim and the lemma. $\square$

Proof of Theorem 3.6: 1) Let I be a Morley sequence of p witnessing the assumption that (p,Q) is ACL-embedded in Q (as in Definition 3.1). By Lemma 3.8 there is a witness $p_{\tilde{b}}\in S(\tilde{b})$ for the internality of p in Q, for some (possibly infinite) tuple of imaginary elements $\tilde{b}\in DCL(p^{\mathcal{C}}\cup Q)$. By ACL-embeddedness of (p,Q) in Q and the fact that $\tilde{b}$ is a tuple of imaginaries, $\tilde{b}\in BDD(Q\cup I)$. Now, $tp(\sigma(\tilde{b})/I\tilde{b})$ is bounded for every $\sigma\in Aut(\mathcal{C}/Q\cup I)$. In particular, p is Q-internal via a disjoint generic parameter modulo p. By Theorem 2.4, there is a set $B\subseteq DCL(p^{\mathcal{C}}\cup Q)$ of imaginaries such that p is controlled by B over Q. Now 2) follows from the following [S3, Theorem 2.9].

Theorem 3.10 *Let T be simple. Let Q be a pseudo-stably-embedded $\emptyset$-invariant set and let $p \in S(\emptyset)$. Suppose there exists a set $B \subseteq DCL(p^{\mathcal{C}} \cup Q)$ of hyperimaginaries with $tp(B) \vdash Lstp(B)$ (e.g. Lstp=stp and we work over $acl^{eq}(\emptyset)$) which controls $p^{\mathcal{C}}$ over Q. Then $Aut(p/Q)$ is type-definable with its action on $p^{\mathcal{C}}$ over $\emptyset$.*

□

4 Corollaries

In this section we show two results that follow from Theorem 3.6. In this section we work in $\mathcal{C}^{eq}$ over a boundedly closed set and T is assumed to be simple.

The first result shows that stable-embeddedness assumptions are sufficient.

Theorem 4.1 *Suppose $p \in S(\emptyset)$ is internal in a quasi-stably-embedded type-definable set Q. Moreover, assume $p^{\mathcal{C}} \cup Q$ is stably-embedded. Then $Aut(p/Q)$ is type-definable with its action on $p^{\mathcal{C}}$.*

Proof: By Theorem 3.6, it is enough to show that (p, Q) is ACL-embedded in Q. Let $\tilde{b} \in DCL^{veq}(p^{\mathcal{C}} \cup Q)$. So, $\tilde{b} = b/E$ where $E = \bigvee_{i \in \mathcal{I}} E_i$ for some directed family $\{E_i\}_{i \in \mathcal{I}}$ of $\emptyset$-definable equivalence relations. Since $p^{\mathcal{C}} \cup Q$ is in particular pseudo-stably-embedded, compactness implies that $b/E_{i^*} \in DCL(p^{\mathcal{C}} \cup Q)$ for some $i^* \in \mathcal{I}$. By Fact 1.1, $p^{\mathcal{C}} \cup Q \subseteq acl(Q \cup I)$ for some Morley sequence I of p (in fact, for every Morley sequence I of p by Theorem 1.2). Thus, since $p^{\mathcal{C}} \cup Q$ is stably-embedded, $DCL(p^{\mathcal{C}} \cup Q) = dcl(p^{\mathcal{C}} \cup Q) \subseteq dcl(acl(Q \cup I)) = acl(Q \cup I) \subseteq ACL(Q \cup I)$. Hence $b/E_{i^*} \in ACL(Q \cup I)$ and therefore $\tilde{b} \in ACL(Q \cup I)$.

□

Now, we discuss the assumption of weak-transitivity of ACL. First, for an $\emptyset$-invariant set U and one-to-finite definable relation $R(x, \bar{y})$ ($\bar{y}$ to x) with parameters, let $acl_R(U) = \{b|\ R(b, \bar{c})$ holds for some tuple $\bar{c}$ from $U\}$.

Definition 4.2 *Let U be an $\emptyset$-invariant set. We say that ACL is weakly-transitive on U with respect to a one-to-finite definable relation $R(x, \bar{y})$ defined over A if $DCL^{eq}(acl_R(U) \cup A) \subseteq ACL^{eq}(U \cup A)$. We say that ACL*

is weakly-transitive on U if it is weakly transitive on U with respect to every one-to-finite relation R (with parameters).

Remark 4.3 Note that the assumption that ACL is weakly-transitive on U is weaker than the assumption that for every small set A,

$$DCL^{eq}(acl^{eq}(U \cup A)) \subseteq ACL^{eq}(U \cup A)$$

and weaker than the assumption that for all small A,

$$ACL^{eq}(ACL^{eq}(U \cup A)) \subseteq ACL^{eq}(U \cup A).$$

Theorem 4.4 *Suppose $p \in S(\emptyset)$ is internal in a definable set Q. Moreover, assume ACL is weakly-transitive on Q. Then $Aut(p/Q)$ is type-definable with its action on $p^{\mathcal{C}}$.*

Proof: By Theorem 3.6, it is enough to show that (p, Q) is ACL-embedded in Q. Indeed, let $\tilde{b} \in DCL^{veq}(p^{\mathcal{C}} \cup Q)$. Again, say $\tilde{b} = b/E$ where $E = \bigvee_{i \in \mathcal{I}} E_i$ for some directed family $\{E_i\}_{i \in \mathcal{I}}$ of $\emptyset$-definable equivalence relations. By Fact 1.1, there exists a one-to-finite definable relation R', defined over a Morley sequence I of p, that covers $p^{\mathcal{C}}$ by Q. So, for some one-to-finite definable relation R, $dcl(acl_R(Q)) \supseteq p^{\mathcal{C}} \cup Q$. Thus $DCL^{veq}(p^{\mathcal{C}} \cup Q) \subseteq DCL^{veq}(acl_R(Q))$. Now, since Q is definable, so is $acl_R(Q)$ (over I). Thus, since $acl_R(Q)$ is, in particular, pseudo-stably-embedded (over I) compactness implies there exists $i^* \in \mathcal{I}$ such that $b/E_{i^*} \in DCL^{eq}(acl_R(Q) \cup I)$. Since ACL is weakly-transitive on Q, $b/E_{i^*} \in ACL^{eq}(Q \cup I)$. In particular, $\tilde{b} \in ACL^{veq}(Q \cup I)$.

□

We draw now another consequence.

Definition 4.5 *We say that the internality of p in Q is based on a set U (possibly not small) if there is a witness p_b for the internality of p in Q and $a \models p_b$ and tuple $\bar{c}$ from Q such that $a \in dcl(b, \bar{c})$ and such that $Cb(Lstp(a, \bar{c}/b)) \in BDD(U)$.*

Remark 4.6 Observe that for every pseudo-stably-embedded set Q we have $BDD^{eq}(Q) = ACL^{eq}(Q)$.

Lemma 4.7 *Let T be any complete theory. Let Q be pseudo-stably-embedded and suppose $A \subseteq ACL^{eq}(Q)$. Then $ACL^{eq}(Q \cup A) = ACL^{eq}(Q)$.*

Proof: Clearly $ACL^{eq}(Q \cup A) \supseteq ACL^{eq}(Q)$. By Remark 4.6, $BDD^{eq}(Q) = ACL^{eq}(Q)$ hence it is enough to prove that if $a \in ACL^{eq}(Q \cup A)$ then $a \in BDD^{eq}(Q)$. Indeed, suppose that $(\sigma_i(a))_{i<\lambda}$ is an unbounded set of distinct Q- conjugates of a with $\sigma_i \in Aut(\mathcal{C}/Q)$. Then for almost all σ_i, $\sigma_i(A)$ is the same, a contradiction. □

Corollary 4.8 *Let Q be a quasi-stably-embedded type-definable set over $\emptyset$. Suppose $p \in S(\emptyset)$ is Q-internal and (p, Q) is ACL-embedded in Q. Then $Aut(p/Q)$ is finite if and only if $p^{\mathcal{C}} \subseteq ACL(Q^{\mathcal{C}})$ (iff $p^{\mathcal{C}} \cap ACL(Q^{\mathcal{C}}) \neq \emptyset$). In particular, if the internality of p in Q is based on $Q^{\mathcal{C}}$ then $Aut(p/Q)$ is finite.*

Proof: If $p^{\mathcal{C}} \not\subseteq ACL(Q^{\mathcal{C}})$ then some (every) element $a \in p^{\mathcal{C}}$ has an infinite orbit over Q. In particular, $Aut(p/Q)$ is infinite. For the other direction suppose $p^{\mathcal{C}} \subseteq ACL(Q^{\mathcal{C}})$. By Theorem 3.6, there is a set $A \subseteq DCL(p^{\mathcal{C}} \cup Q^{\mathcal{C}})$ that controls p over Q. By our assumption and Lemma 4.7, for some Morley sequence I of p, $DCL(p^{\mathcal{C}} \cup Q^{\mathcal{C}}) \subseteq ACL(Q^{\mathcal{C}} \cup I) \subseteq ACL(Q^{\mathcal{C}})$. Hence $A \in BDD(Q^{\mathcal{C}})$ (where A is considered as a single hyperimaginary). Hence the size of $Aut(p/Q)$ is bounded. Since $Aut(p/Q)$ is type-definable it is finite. Now, if the internality of p in Q is based on Q, then there exists $a \models p$, hyperimaginary $\tilde{b} \in BDD(Q)$, and tuple $\bar{c}$ from Q such that $a \in acl(\tilde{b}, \bar{c})$. Thus $a \in BDD(Q)$ and thus $a \in ACL(Q)$. Thus $p^{\mathcal{C}} \subseteq ACL(Q)$. □

Remark 4.9 In the stable case (when working over $acl(\emptyset)$) if the internality of p in Q is based on $\emptyset$ then $Aut(p/Q) = \{id\}$.

References

[H1] E.Hrushovski, Unidimensional theories are superstable, the Annals of Pure and Applied Logic 50, 1990, pgs 117–138. bibitem[H2]Hrgrp E. Hrushovski, unpublished notes on Picard-Vessiot theory.

[S1] Z.Shami, Definability in low simple theories, The Journal of Symbolic Logic, Volume 65, Number 4, Dec. 2000, pgs. 1481–1490.

[S2] Z.Shami, Internality and interpretable automorphism groups in simple theories, to appear in Annals of Pure and Applied Logic.

[S3] Z.Shami, Coordinatization by binding groups and unidimensionality in simple theories, submitted.

[SW] Z.Shami and F.Wagner, On the binding group in simple theories, The Journal of Symbolic Logic, Volume 67, Number 3, September 2002.

[W] Frank O. Wagner, Simple Theories, Academic Publishers, Dordrecht, The Netherlands, 2000.

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