Final Exam, Math 4LT3/6LT3
Apr. 14 – 16, 2008

This final exam is due by 4:30 p.m., April 16, 2008. You may send your solutions to me by email or hand your exam in to the main department office by that time. You may use any literature you wish to solve the problems on this final but you must cite your sources. You may not collaborate with anyone in order to solve the questions on this exam. Throughout the exam, assume that we are working in ZFC unless the question explicitly says otherwise.

1. Fix a set X of regular, uncountable cardinality λ. We say that $\langle X_\alpha : \alpha < \lambda \rangle$ is a filtration of X if:

   (a) $X_\alpha \subseteq X_\beta \subseteq X$ for all $\alpha < \beta < \lambda$;

   (b) for limit ordinals $\delta < \lambda$, $X_\delta = \bigcup\{X_\alpha : \alpha < \delta\}$ and $X = \bigcup\{X_\alpha : \alpha < \lambda\}$; and

   (c) $|X_\alpha| < \lambda$ for all $\alpha < \lambda$.

   Prove that if $\langle A_\alpha : \alpha < \lambda \rangle$ and $\langle B_\alpha : \alpha < \lambda \rangle$ are two filtrations of X then $C = \{\alpha < \lambda : A_\alpha = B_\alpha\}$ is closed and unbounded i.e. (unbounded) for every $\alpha < \lambda$, there is $\beta \in C$ such that $\beta > \alpha$ and (closed) if $U \subset C$, $|U| < \lambda$ then $\bigcup U \in C$. Hint: First show that for every $\alpha < \lambda$ there is a $\beta > \alpha$ such that $A_\alpha \subseteq B_\beta$.

2. Remember the inductive definitions:

   (a) $\aleph_0 = \beth_0 = \omega$;

   (b) for successor ordinals $\alpha + 1$, $\aleph_{\alpha+1} = \aleph_\alpha^+$ and $\beth_{\alpha+1} = |\mathcal{P}(\beth_\alpha)|$; and

   (c) for limit ordinals δ, $\aleph_\delta = \bigcup\{\aleph_\alpha : \alpha < \delta\}$ and $\beth_\delta = \bigcup\{\beth_\alpha : \alpha < \delta\}$.

   Prove

   (a) $\alpha \leq \aleph_\alpha \leq \beth_\alpha$ for every ordinal α.

   (b) For arbitrarily large α, $\aleph_\alpha = \beth_\alpha$ i.e. for every ordinal β there is $\alpha > \beta$ such that $\aleph_\alpha = \beth_\alpha$.

   (c) For arbitrarily large α, $\alpha = \aleph_\alpha$.

3. (a) Prove that for every countable ordinal γ there is an order-preserving function $f_\gamma : \gamma \to \mathbb{R}$ i.e. if $\alpha < \beta < \gamma$ then $f_\gamma(\alpha) < f_\gamma(\beta)$.
   (b) Prove that there is no order-preserving $f : \omega_1 \to \mathbb{R}$.

4. In this course, the axiom of infinity (Inf) and the regularity axiom (Reg) played important roles. The purpose of this question is to examine the strength of these axioms. Let NoInf be all the axioms of ZF except for Inf and let NoReg be all the axioms of ZF except for Reg.

   In this question, I refer to Chapter 8 where the sequence of sets $V(\alpha)$ was defined for all ordinals α. The definition of this hierarchy can be accomplished in NoReg (in Chapter 8, Henle is implicitly assuming ZF but to use Theorem 6.14 you don't need Reg.)

   (a) Show that NoReg implies Con(ZF). To show this, we suppose that we have a universe of sets U which satisfies NoReg. Remember, these sets will not necessarily be well-founded so for instance there could be $x \in U$ satisfying $x \in x$. Nevertheless, in U, define V as in Chapter 8 and verify that ZF holds in V. In fact, Katie's proof of Theorem 8.10 works for all axioms as written except for Regularity and Replacement so just verify that these are true in V.
   (b) Show that ZF implies Con(NoInf + $\neg$Inf). Hint: Show that if U is any universe of sets satisfying ZF then $V(\omega)$ in U satisfies NoInf + $\neg$Inf.