Math 2S03 Extra practice problems for the final exam 04–06–2011

Part I. Review problems from the textbook.
Chapter 1 p.12: # 1, 2, 3
Chapter 2, p.31: # 4, 6, 27*
Chapter 3 p.54: # 7, 22, 26, 29
Chapter 4, p.72: # 11,13, 19, 20, 29
Chapter 5 p.94: # 5, 6, 15
Chapter 6 p.125: # 4, 7, 17, 26
Chapter 7 p.147: # 18, 23
Chapter 9 p.181: # 10, 21, 29, 30
Chapter 10 p. 209: # 5, 8, 9, 18, 26

1. Suppose $F = F_2 = \{0, 1\}$ is the field with two elements and $P_n(F)$ is the vector space of polynomials of degree at most n. Define $T : P_5(F) \to P_5(F)$ by sending $T(f(x)) = f(x + 1)$.
(a) Show that T is a linear transformation.
(b) Is T is an involution? Why or why not?
(c) Find the matrix representative of T with respect to the standard basis $\{1, x, x^2, \ldots, x^5\}$.

2. Suppose $L : V \to V$ is a linear transformation of a finite dimensional vector space over a field F different from the identity, i.e. with $L \neq I_V$. Given that L is an involution, what is the minimum polynomial of L? What does one need to assume about the underlying field F in order to ensure that L diagonalizable?

3. Suppose X is a set with at least two elements and F is some field. Let V denote the vector space $\mathcal{F}(X, F) = \{f : X \to F\}$ of all functions from X to F. Choose distinct elements $x, y \in X$ and define W to be the set of all functions $f \in V$ satisfying $f(x) = f(y)$. Show that W is a subspace of V.

4. Suppose V is a real inner product space and $T : V \to V$ is a self-adjoint transformation. Show that $\ker(T) = \operatorname{im}(T)^{\perp}$.

5. A matrix $A \in M_6(F)$ has characteristic polynomial $c_A(x) = x^6$ and minimum polynomial $p_A(x) = x^2$. Determine all possible Jordan forms for A and the geometric multiplicities of all eigenvalues.

6. Suppose V is a vector space and $L : V \to V$ is a linear transformation. Show that if $U \subseteq V$ is an L-invariant subspace, then the annihilator $\operatorname{Ann}(U)$ of U is an L^*-invariant subspace of V^*.

7. (a) Determine e^{tA} for

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

and find the general solution to the linear system of differential equations

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \qquad \text{for} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

8. Recall that a linear transformation $T : V \to V$ is called *cyclic* if there exists $v \in V$ (called a cyclic vector) such that $\{v, T(v), T^2(v), \ldots, T^j(v) \ldots\}$ spans V.
(a) Assuming $\dim V = n$ has dimension n, show that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is a basis for V.
(b) Writing $T^n(v) = a_0 v + a_1 T(v) + \cdots + a_{n-1} T^{n-1}(v)$ for $a_0, a_1, \ldots, a_{n-1} \in F$, show that T is invertible if and only if $a_0 \neq 0$. (Hint: show that $\det(T) = a_0$).
(c) In a previous assignment, you showed that T has characteristic and minimum polynomial $c_T(x) = m_T(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$. In case $a_0 \neq 0$, show that one can solve for T^{-1} as a polynomial in T via the formula:

$$T^{-1} = (a_0)^{-1} \left(T^{n-1} - a_{n-1} T^{n-2} - \cdots - a_1 T \right).$$

(d) Using this formula, determine the inverse to the (cyclic) matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

9*. In this problem, we will exhibit an infinite dimensional vector space V which is not isomorphic to its dual V^*. To start, suppose F is a field and let $V = F[x]$ be the vector space of polynomials over F. For any $a \in F$, define $\phi_a : V \to F$ by $\phi_a(f(t)) = f(a)$.
(a) Show that ϕ_a is a linear functional, i.e. that $\phi_a \in V^*$.
(b) Show that if $a_1 \ldots, a_n \in F$ are n distinct elements, then $\{\phi_{a_1}, \phi_{a_2}, \ldots, \phi_{a_n}\}$ is a linearly independent subset of V^*.
(Hint: For $n = 2$, consider the polynomials $f_1(x) = x - a_2$ and $f_2(x) = x - a_1$; for $n = 3$, take $f_1(x) = (x - a_2)(x - a_3)$, $f_2(x) = (x - a_1)(x - a_3)$ and $f_3(x) = (x - a_2)(x - a_3)$; for $n = 4$ ...)
(c) More generally, show that for any subset $A \subset F$, the set $\{\phi_a \mid a \in A\}$ is a linearly independent subset of V^*.
(Hint: a linear combination from the set $\{\phi_a \mid a \in A\}$ is by definition a *finite* sum of the form $\sum_{i=1}^n c_i \phi_{a_i}$.)
(d) Now suppose F is uncountable (eg. $\mathbb{R}$ or $\mathbb{C}$) and using part (c), show that the dual space $(F[x])^*$ admits an uncountable linearly independent set. This shows that $(F[x])^*$ is not isomorphic to $F[x]$, since $F[x]$ admits a countable basis, namely the standard basis.
(e) Does the same argument work for polynomial rings over countable fields, say $\mathbb{Q}[x]$ and its dual? Why or why not?

*Starred problems are meant to be challenging.