Math 2S03 **Assignment 7** **due 04/05/2011**

Part I. Homework problems from textbook.
These problems are assigned but will not be marked.
Chapter 7 p.147: # 4, 9, 15, 19, 24, 28
Chapter 9 p.181: # 18, 19, 20, 21
Chapter 10 p. 209: # 5, 18

Part II. Please write up and hand in solutions to the following problems.

1. Suppose V is a complex inner product space and $T : V \to V$ is a self-adjoint transformation, and $v \in V$ is an eigenvector with eigenvalue λ. Show that $v^{\perp}$ (the orthogonal complement of v) is a T-invariant subspace.

2. Let $w = (1, 2, 3, 1) \in \mathbb{R}^4$ and find an orthogonal basis for $w^{\perp}$.

3. Suppose $S, T \subseteq V$ are subsets of a real inner product space V. Prove:
(a) $S \subseteq S^{\perp\perp}$.
(b) $S \subseteq T \Rightarrow T^{\perp} \subseteq S^{\perp}$
(c) $S^{\perp} = \mathrm{span}(S)^{\perp}$.

3. Using the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$, apply the Gram-Schmidt to the polynomials $\{1, x, x^2\}$ to produce an orthonormal family. (The polynomials obtained from this procedure are the called *Legendre* polynomials.)

4. Use the Cauchy-Schwarz inequality on the appropriate inner product space to give a bound for the integral

$$\int_0^{\pi/2} \sqrt{x \sin(x)}dx.$$

5. (a) If $W \subseteq V$ is a subspace, then show that $V = W \oplus W^{\perp}$.
(b) If, in addition, V is finite dimensional, then $W^{\perp\perp} = W$.

6. Prove the following statements:
(a) $P \in M_n(\mathbb{R})$ is orthogonal if and only if P^t is orthogonal.
(b) If $P \in M_n(\mathbb{R})$ is orthogonal, then P^{-1} is orthogonal.
(c) If $P, Q \in M_n(\mathbb{R})$ are both orthogonal, then their product PQ is also orthogonal.

7. Suppose V is a complex inner product space. Prove that

$$\langle u, v \rangle = \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2\right).$$

8. (a) Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculate e^{tA} explicitly by finding a matrix P so that $P^{-1}AP$ is in Jordan form.
(b) Use this to find the general form of the solution to the system of differential equations

$$\begin{aligned} dx/dt &= x + 2y + 3z \\ dy/dt &= y + 2z \\ dz/dt &= z \end{aligned}$$