SAMPLE PURE MATH PRELIMINARY EXAM

A. Core Material

Answer four of the following six questions.

Problem A.1. Let V be a real vector space with a positive definite inner product $\langle\ ,\ \rangle$, and let $v_1, \cdots, v_n$ be vectors in V. Show that the $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} = \langle v_i, v_j\rangle$ is invertible if and only if the vectors $v_1, \cdots, v_n$ are linearly independent.

Problem A.2. Let V be the vector space of 2×2 matrices over the reals and suppose $A = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix}$. Define the linear transformation $T : V \to V$ by $T(X) = AXB$. Calculate the trace and the determinant of T.

Problem A.3.
(a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $[a, b]$ is a compact interval, then there exists $x_0 \in [a, b]$ so that $\sup_{x\in[a,b]} f(x) = f(x_0)$.
(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable on (a, b), and f attains its maximum at $x_0 \in (a, b)$, then $f''(x_0) \leq 0$.

Problem A.4. Define $\mathscr{M} = [1, \infty)$ with distance function

$$d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|.$$

(a) Prove that $(\mathscr{M}, d)$ defines a metric space.
(b) Show that $\mathscr{M}$ is a bounded metric space but not a compact metric space.
(c) Is $\mathscr{M}$ complete? Prove your assertion.

Problem A.5. Find a one-to-one conformal mapping from the disk $D_1 = \{z \in \mathbb{C},\ |z| < 1\}$ onto $D_2 = \{w \in \mathbb{C},\ |w + 1| < 1\}$ with $f(0) = -1/2$.

Problem A.6. Use contour integration to evaluate

$$\int_0^\infty \frac{\ln x}{4 + x^2}\, dx.$$

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B. Pure Math Topics

Answer three of the following four questions.

Problem B.1. Determine all groups of order 8.

Problem B.2. Let $p < q < r$ be prime numbers and let G be a group of order pqr. Show that G is not simple.

Problem B.3. Let p be a prime number and R a ring with identity containing exactly p^2 elements. Prove that R is commutative.

Problem B.4. (a) Let K be a field, $f(X)$ a non-zero polynomial in $K[X]$. Show that the following are equivalent.

  (i) The ideal $(f(X))$ is prime.
  (ii) The ideal $(f(X))$ is maximal.
  (iii) The polynomial $f(X)$ is irreducible.

(b) Let $f(X) = X^3 - 3X^2 + 6X - 7$. Prove that $\mathbb{Q}[X]/(f(X))$ is a field.