ADDITIONAL REVIEW PROBLEMS FOR THE PURE MATH PRELIMINARY EXAM

A. Group Theory

Problem A.1 (4/1995)**.** (a) State the three Sylow theorems.
(b) Show that there is no simple group of order 28.

Problem A.2 (4/1995)**.** Prove there is no simple group of order 56.

Problem A.3 (4/1995)**.** Determine all groups of order 8.

Problem A.4 (4/1995)**.** Show that any group of order p^n (p prime) is solvable.

Problem A.5 (4/1995)**.** Let G be a group, H and K normal subgroups with $H \subset K$. Show that there is a canonical isomorphism

$$\frac{G/H}{K/H} \cong \frac{G}{K}.$$

Problem A.6 (4/1995)**.** Show that a *finite* subgroup of the multiplicative group F^* of a field K is cyclic.

Problem A.7 (4/1995)**.** (a) Suppose G is a group and $H \subset G$ is a subgroup of index two. If H is central, show that G is abelian. (Recall that H central means that H is contained in the center of G.)
(b) Find all finite groups G whose automorphis group has order two.

Problem A.8 (12/1997)**.** Let c be a real constant (the speed of light!) and let v be a real variable such that $-c < v < c$. Write $A(v)$ for the 2×2 matrix

$$A(v) = \lambda(v) \begin{bmatrix} 1 & -v \\ -v/c^2 & 1 \end{bmatrix},$$

where $\lambda(v) = (1 - v^2/c^2)^{-1/2}$. (Note that $1 - v^2/c^2 = \det \begin{bmatrix} 1 & -v \\ -v/c^2 & 1 \end{bmatrix}$.)
(a) Show that their matrix product satisfies $A(v_1)A(v_2) = A(v_3)$ where $v_3 = \dfrac{v_1 + v_2}{1 + v_1 v_2/c^2}$.
(b) Use part (a) to show that matrix multiplication turns the set $G = \{A(v) \mid -c < v < c\}$ into a group.
(Note: G is the group of symmetries of two-dimensional special relativity and is called the *Lorentz* group.)

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Problem A.9 (12/1997)**.** Let G by a finite group, $H \subset G$ a normal subgroup and p a prime. Suppose S is a Sylow p-subgroup of G.
(a) Prove that the intersection $S \cap H$ is a Sylow p-subgroup of H.
(b) Prove that HS/H is a Sylow p-subgroup of G/H.
(Hint: Compare the orders of these groups and use the Second Isomorphism Theorem.)

Problem A.10 (4/1998)**.** Let G be a finite group, H a normal subgroup and P a p-Sylow-subgroup of H. Show that

$$G = H \cdot N_G(P),$$

and that $[G : H]$ divides $|N_G(P)|$. (Here $N_G(P)$ denotes the normalizer of P in G.)

Problem A.11 (4/1998)**.** Let G be a non-abelian group of order p^3, p a prime. Show that the center $Z(G)$ has precisely p elements and that

$$G/Z(G) \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}.$$

Problem A.12 (8/1998)**.** Let G be a finite group and let H be a subgroup of index n. Show that there exists a normal subgroup N of G contained in H, such that

$$[G : N] \leq n!$$

Problem A.13 (8/1998)**.** Let S_n denote the symmetric group on n letters. Show that the center $Z(S_n)$ of S_n is trivial for $n \geq 3$.

Problem A.14 (12/1998)**.** Let G be a group of order 44 which has a normal subgroup of order 4. Prove that G is an abelian group.

Problem A.15 (12/1998)**.** Let G be an abelian group generated by at most n elements. Show that each subgroup of G is again generated by at most n elements.

Problem A.16 (12/1999)**.** Prove that every nilpotent group is solvable. Give an example of a solvable group which is not nilpotent.

Problem A.17 (12/1999)**.** Suppose that p is a prime and G is a finite group, such that the index of every proper subgroup of G is divisible by p. Prove that G has p-power order.

Problem A.18 (4/2000)**.** Let G be a group and assume that G does not contain a subgroup of index 2. Prove that every subgroup of index 3 is normal.

Problem A.19 (4/2000)**.** Let A, B, C be finite abelian groups. Prove that

$$A \times B \cong A \times C$$

implies that

$$B \cong C.$$

Provide an example to show that the conclusion may be wrong if not all 3 groups A, B and C are assumed to be finite.

Problem A.20 (12/2000)**.** Suppose that G is a finite group of order $2907 = 3^2 \cdot 17 \cdot 19$. Use Sylow's Theorems to prove that G has a normal subgroup either of order 17 or of order 19.

Problem A.21 (12/2000)**.** Suppose that H is a subgroup of index n of the group G. Prove that there is a normal subgroup K of G such that $K \subset H$ and $[G : K] \leq n!$.

Problem A.22 (12/2001)**.** Let G be a finite group.
(a) Suppose H, N are normal subgroups of G, and $|H|$ and $|N|$ are relatively prime. Prove that $xy = yx$ for all $x \in H$ and $y \in N$ and hence that $H \times N \cong HN$.
(b) Suppose all the Sylow subgroups of G are normal. Prove that G is the direct product of its Sylow subgroups.

Problem A.23 (12/2001)**.** Let G be group of order $2 \cdot 13 \cdot 23$. Prove that at least one of the Sylow subgroups of G is normal.

Problem A.24 (8/2002)**.** Let G be the subgroup of $GL_2(\mathbb{C})$ (the group of invertible 2×2 matrices over the complex numbers) generated by the matrices $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, where $i = \sqrt{-1}$.
(a) Is G finite? If so, what is the order of G?
(b) Determine the center of G. (List the elements of the center of G explicitly.)
(c) Is G an abelian group?

Problem A.25 (8/2002)**.** Let G be a group of order $2552 = 8 \cdot 11 \cdot 29$. Use Sylow's Theorems to prove that G has a normal subgroup of order 11 or of order 29.

Problem A.26 (5/2003)**.** Show that a finite group is cyclic if and only if it is not equal to the union of its proper subgroups.

Problem A.27 (5/2003)**.** Let $p < q < r$ be prime numbers and let G be a group of order pqr. Show that G is not simple.

B. Rings & Modules

Problem B.1 (4/1998)**.** Let R be a commutative ring with 1. Show: If the localization $R_{\mathfrak{m}}$ of R with respect to each maximal ideal $\mathfrak{m}$ of R contains no nilpotent elements, then the same is true for R.

Problem B.2 (4/1998)**.** Let I be an ideal in a commutative ring R with 1. Show: If I is a free R-module, then I is a principal ideal. Is the converse true as well?

Problem B.3 (8/1998)**.** Let R denote an integral domain. Assume that there is a function $f : R \to \mathbb{N} \cup \{0\}$ with the following properties:
(i)$f(x) = 0 \iff x = 0$.
(ii)$f(xy) = f(x)f(y)$.
(iii) If $x, y \in R$ satisfy $0 < f(x) \leq f(y)$, then either x divides y or there exist $s, t \in R$, such that $0 < f(sx - ty) < f(y)$.

Show that R is a Principal Ideal Domain.

Problem B.4 (8/1998)**.** Let R be a commutative ring with 1, let M be a Noetherian R-module, and let $\phi : M \to M$ be a homomorphism. Show that there exists $n \geq 1$, such that

$$\ker \phi^n \cap im\, \phi^n = (0).$$

Problem B.5 (12/1998)**.** Give a sketch of a proof of the following statement:

> A finitely generated module A over a principal ideal domain R is equal to a direct sum of cyclic modules.

Problem B.6 (12/1998)**.** Let R be a ring. A left ideal L is called *maximal* if $L \neq R$, and I is a left ideal $L \subset I \subset R$, then either $I = L$ or $I = R$. Let N be the intersection of *all* maximal left ideals in R.

(a) Show that N is a left ideal of R.

(b) Assume S is a simple R module and show that $NS = 0$. (Consider the kernel of the homomorphism $R \to S$ of left R modules given by $a \mapsto ax$ where x is any nonzero element of S.)

(c) Conversely if $a \in R$ has the property that $aS = 0$ for all simple R modules S, show that $a \in N$. (Consider the simple R module R/L where L is a maximal left ideal.)

(d) Use part (c) to show that N is a two sided ideal.

Problem B.7 (12/1999)**.** Let R be a finite commutative ring with 1. Prove that every prime ideal in R is maximal.

Problem B.8 (12/1999)**.** Let M be a finitely generated module over the principal ideal domain R. Let $\operatorname{Tor}(M)$ denote the set of torsion elements of M. Show that $\operatorname{Tor}(M)$ is a submodule of M, and describe the structure of $M/\operatorname{Tor}(M)$.

Problem B.9 (4/2000)**.** Let R be a ring with identity, and let u be an element of R with a right inverse. Prove that the following conditions on u are equivalent:

(a) u has more than one right inverse,

(b) u is a zero divisor,

(c) u is not a unit.

Problem B.10 (4/2000)**.** Let p be a prime number and R a ring with identity containing exactly p^2 elements. Prove that R is commutative.

Problem B.11 (12/2000)**.** Let $\mathbb{K}$ be a field. Define

$$A = \left\{ \begin{bmatrix} x & -y \\ 5y & x+y \end{bmatrix} \mid x, y \in \mathbb{K} \right\}$$

(a) Show that A is a commutative subring of the ring of 2×2 matrices over $\mathbb{K}$.

(b) Prove that if $u^2 + 19 \neq 0$ for all $u \in \mathbb{K}$ then A is a domain.

(c) Show that if $\operatorname{char} \mathbb{K} = 19$ then A has a non-zero nilpotent element.

Problem B.12 (12/2000)**.** We have the following inclusions among classes of commutative rings with identity:

$$\text{fields} \subset \text{P.I.D.'s} \subset \text{U.F.D's} \subset \text{integral domains}$$

Give examples to show that each of these inclusions is proper.

Problem B.13 (12/2001)**.** Let R be a commutative ring with 1, S a multiplicative subset of R (that is, $1 \in S$ and for all $x, y \in S$, $xy \in S$). Define

$$S^{-1}R := \{(r, s) : r \in R, s \in S\}.$$

Define addition and multiplication on $S^{-1}R$ in the obvious way:

$$\begin{aligned} (r, s) + (r', s') &= (rs' + r's, ss') \\ (r, s)(r', s') &= (rr', ss') \end{aligned}$$

(a) Prove that addition and multiplication are well-defined, hence $S^{-1}R$ is a ring.
(b) Prove that if R is a principal ideal domain then $S^{-1}R$ is a principal ideal domain.
(c) Let $\mathscr{P}$ be a prime ideal in R and let $S = R \setminus \mathscr{P}$. Verify that S is multiplicative, and prove that $S^{-1}R$ has a unique maximal ideal.

Problem B.14 (12/2001)**.** (a) Let K be a field, $f(X)$ a non-zero polynomial in $K[X]$. Show that the following are equivalent.
(i) The ideal $\langle f(X) \rangle$ is prime.
(ii) The ideal $\langle f(X) \rangle$ is maximal.
(iii) The polynomial $f(X)$ is irreducible.

(b) Let $f(X) = X^3 - 3X^2 + 6X - 7$. Prove that $\mathbb{Q}[X]/\langle f(X) \rangle$ is a field.

Problem B.15 (4/2002)**.** Let R be an integral domain. Prove that R is a field if every principal ideal in R is a prime ideal.

Problem B.16 (4/2002)**.** Let R be an integral domain, and let $R[X]$ denote the polynomial ring in one variable over R. Prove that a polynomial $f(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n \in R[X]$ is a unit in $R[X]$ if and only if a_0 is a unit in R and $a_1 = a_2 = \cdots = a_n = 0$.

Problem B.17 (5/2003)**.** Let R be a commutative ring with 1.
(a) Give the definitions of (i) a prime ideal of R, and (ii) a maximal ideal of R.

(b) Suppose for every $r \in R$ there exists an integer $n > 1$ (depending on r) such that $r^n = r$. Show that every prime ideal of R is maximal.

Problem B.18 (5/2003)**.** (a) Give the definitions of (i) Principal Ideal Domain (PID), and (ii) Unique Factorization Domain (UFD).

(b) Which of the following rings are PIDs? Which are UFDs?
(i) $\mathbb{Z}[x]$,
(ii) $\mathbb{Q}[x, y]$,
(iii) $\mathbb{Z}[\sqrt{-1}]$,
(iv) $\mathbb{Z}[2\sqrt{2}]$.