REVIEW PROBLEMS FOR THE CORE PART OF THE PRELIMINARY EXAM

A. Linear Algebra

Problem A.1 (4/1995)**.** Describe up to similarity all real 3×3 matrices A for which $A^2 = A^3$.

Problem A.2 (4/1995)**.** Let V be the Euclidean space $\mathbb{R}^n$ with the standard inner product $x \cdot y$.
(a) If $\phi_1, \phi_2, \ldots, \phi_k \in \operatorname{End} V$, show that

$$x * y := \sum_{i=1}^{k} \phi_i(x) \cdot \phi_i(y)$$

is also an inner product on V.
(b) Let $G = \{\phi_1, \phi_2, \ldots, \phi_k\} \subset \operatorname{Aut} V$ be any finite group of linear transformations on V. Show that G is a subgroup of the orthogonal group of $x * y$.
(c) Show that G is conjugate in $\operatorname{Aut} V$ to a subgroup of the "usual" orthogonal group

$$O(n) = \{\psi \in \operatorname{Aut} V \mid \psi(x) \cdot \psi(y) = x \cdot y\}.$$

Problem A.3 (4/1995)**.** Let A be a real $m \times n$ matrix, viewed as a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$, and let A^t be its transpose, viewed as a linear transformation $A : \mathbb{R}^m \to \mathbb{R}^n$.
(a) Show that the subspace $\operatorname{Im} A^t$ and $\ker A$, the image of A^t and the kernel of A, are orthogonal with respect to the usual inner product in $\mathbb{R}^n$, and that $\mathbb{R}^n$ is the direct sum

$$\mathbb{R}^n = \operatorname{Im} A^t \oplus \ker A.$$

(b) Deduce that if the linear system of equations $A\mathbf{x} = \mathbf{b}$ has a solution, then is has a unique solution of the form $\mathbf{x} = A^t\mathbf{y}$.

Problem A.4 (12/1997)**.** Let V be an inner product space and $P : V \to V$ a projection (i.e. $P^2 = P$). Show that P is an orthogonal projection if and only if P is self-adjoint.

Problem A.5 (12/1997)**.** Let $T : V \to V$ be a linear transformation on an n-dimensional vector space V and assume that $\ker(T^{n-1}) \neq \ker(T^n)$. Show that for each k with $1 \le k \le n$, the dimension of $\ker(T^k)$ is precisely k. Deduce that T is nilpotent.

Problem A.6 (4/1998)**.** Let V be a finite-dimensional real inner product space, and let $a, b \in V$ be two vectors of the same length. Show that there exists an isometry $\sigma : V \to V$, such that $\sigma(a) = b$.

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Problem A.7 (4/1998)**.** Let V be a finite-dimensional complex vector space and let $N : V \to V$ be a nilpotent linear transformation. Show that the total number of Jordan blocks in the Jordan canonical form for N is equal to $\dim\ker(N)$.

Problem A.8 (8/1998)**.** Let A be a symmetric $n \times n$-matrix with real coefficients. Show that there exists a matrix S with complex coefficients, such that

$$A = SS^T.$$

Problem A.9 (8/1998)**.** Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let

$$b : V \times V \to \mathbb{F}$$

be a bilinear form (not necessarily symmetric). Let

$$U = \{\mathbf{x} \in V \mid b(\mathbf{x}, \mathbf{y}) = 0 \text{ for all } \mathbf{y} \in V\}$$

Problem A.10 (12/1998)**.** Show that

$$\det \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_k \\ x_1^2 & x_2^2 & \dots & x_k^2 \\ \vdots & \vdots & & \vdots \\ x_1^{k-1} & x_2^{k-1} & \dots & x_k^{k-1} \end{bmatrix} = \prod_{i<j}(x_i - x_j).$$

Problem A.11 (12/1998)**.** Let A and B be symmetric $n \times n$ matrices with real entries such that $AB = BA$. Show that A and B can be simultaneously diagonalized.

Problem A.12 (12/1998)**.** Let T be a linear transformation on a finite dimensional vector space V. Show that T is nilpotent if and only if all the eigenvalues of T equal zero.

Problem A.13 (4/2000)**.** Let V be a finite-dimensional vectorspace over some field k, and let $p : V \to V$ be an endomorphism of V, such that $p^2 = p$.
(a) Show that $V = \ker p \oplus \operatorname{im} p$.
(b) Show that there exists a basis of V, such that the matrix of p with respect to this basis has the form

$$\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}.$$

Problem A.14 (4/2000)**.** Let V be a real vector space with a positive definite inner product $\langle\ ,\ \rangle$, and let $v_1, \cdots, v_n$ be vectors in V. Show that the $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} = \langle v_i, v_j\rangle$ is invertible if and only if the vectors $v_1, \cdots, v_n$ are linearly independent.

Problem A.15 (12/2000)**.** Let V be an n-dimensional vector space over a field $\mathbb{K}$. A linear transformation $T : V \to V$ is said to be *nilpotent of index* $k \geq 1$ if $T^k = 0$ and $T^{k-1} \neq 0$.
a) Prove that if T is nilpotent of index k then $k \leq n$.
b) Give an example of a linear transformation $T : V \to V$ which is nilpotent of index n.

Problem A.16 (12/2000)**.** Let V be an n-dimensional vector space over a field $\mathbb{K}$ where $n > 1$. Let $f : V \to \mathbb{K}$ be a given surjective linear transformation. For each $v \in V$ define an endomorphism of V, $F_v : V \to V$, by $F_v(x) = x - f(x)v$.
(a) Find the determinant of F_v. (Hint: Determine all the eigenvalues of F_v.)
(b) Suppose that $\mathbb{K} = \mathbb{R}$ and that V is a real inner product space. Determine those $v \in V$ for which F_v is an orthogonal transformation.

Problem A.17 (12/2001)**.** Consider the matrix $A = \begin{bmatrix} -28 & 18 \\ -54 & 35 \end{bmatrix}$.
(a) Find a real matrix B such that $B^3 = A$.
(b) How many distinct complex matrices X satisfy $X^3 = A$?

Problem A.18 (12/2001)**.** Let n be a positive integer and V_n the complex vector space of $n \times n$ complex matrices. For A, $B \in V_n$, define $\langle A, B\rangle = tr(AB^*)$, where B^* denotes the conjugate transpose of B and $tr(X)$ denotes the trace of a matrix X.
(a) Show that $\langle \cdot, \cdot \rangle : V_n \times V_n \to \mathbb{C}$ as above defines an inner product on V_n.
(b) Find the orthogonal complement of the subspace of diagonal matrices.

Problem A.19 (8/2002)**.** Determine the Jordan Canonical Form of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 6 \\ 0 & 0 & 4 \end{bmatrix}.$$

Problem A.20 (8/2002)**.** Let V be a finite dimensional vector space and $S, T : V \to V$ a pair of linear transformations. Prove that

$$\dim \ker(ST) \leq \dim \ker S + \dim \ker T.$$

Problem A.21 (5/2003)**.** Let V be the vector space of 2×2 matrices iver the reals and suppose $A = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix}$. Define the linear transformation $T : V \to V$ by $T(X) = AXB$. Calculate the trace and the determinant of T.

Problem A.22 (5/2003)**.** Consider the matrix $A = \begin{bmatrix} -8 & -6 & -1 \\ 18 & 13 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.
(a) Find a real matrix B such that $B^2 = A$.
(b) How many distinct real matrices X satisfy $X^2 = A$?

B. Basic Real Analysis

Problem B.1 (12/1994)**.** Let

$$f_n(x) = \frac{nx}{1 + n^p x^2}, \qquad x \in \mathbb{R}, \quad p > 0.$$

(a) For which values of $p > 0$ does $f_n(x) \to 0$ pointwise on $\mathbb{R}$?
(b) For which values of $p > 0$ does $f_n(x) \to 0$ uniformly on $\mathbb{R}$?

Problem B.2 (12/1994)**.** Suppose $\{f_n\} \subset C^1([0,R]),\ 0 < r < \infty$, with $f_n(0) = 0$ for all $n = 0, 1, \dots,$ and

$$\sup_n \int_0^R |f_n'(x)|^2\,dx \le 4.$$

(a) Show that the sequence $\{f_n\}$ admits a uniformly convergent subsequence.
(b) Show by example that this conclusion is no longer valid if $R = \infty$.

Problem B.3 (12/1994)**.** (a) Suppose $f \in C^2([a,\infty))$ for $a \in \mathbb{R}$ fixed, with

$$\begin{cases} M_0 = \sup_{x\ge a} |f(x)| < +\infty, \\ M_2 = \sup_{x\ge a} |f''(x)| < +\infty. \end{cases}$$

Show that $M_1 = \sup\limits_{x\ge a} |f'(x)| < +\infty$ and that $M_1^2 \le 4M_0M_2$.

(b) Suppose that $f \in C^2([0,\infty))$, $|f(x)| \le 1$ for all $x \in [0,\infty)$, and $f''(x) \to 0$ as $x \to \infty$. Show that $f'(x) \to 0$ as $x \to \infty$.

Problem B.4 (5/1996)**.** (a) State and prove the mean value theorem.
(b) Give an example to show that the mean value theorem can fail for vector-valued functions.

Problem B.5 (5/1996)**.** Suppose that K is compact and that E is a dense subset of K. If $\{g_i\}_{i=1}^\infty \subset C(K)$ is equicontinuous and $\{g_i(x)\}_{i=1}^\infty$ converges for each $x \in E$, show that $\{g_i\}_{i=1}^\infty$ converges uniformly on K.

Problem B.6 (5/1996)**.** Define $s_1 = \sqrt{14}$ and $s_{n+1} = \sqrt{14 + \sqrt{s_n}}$ for $n \ge 1$. Show that $\lim\limits_{n\to\infty} s_n$ exists and equals 4. (Hint: Show that $s_{n+1} \ge s_n$ for all $n \ge 1$.)

Problem B.7 (5/1996)**.** Let X be a metric space.
(a) If $\{x_n\}_{n=1}^\infty$ converges to $x_0 \in X$, show that $\{x_n\}_{n=1}^\infty \cup \{x_0\}$ is compact.
(b) If $\{K_n\}_{n=1}^\infty$ is a decreasing sequence of non-empty compact subsets of X, i.e. $K_n \supset K_{n+1} \ne \varnothing$ for $n \ge 1$, prove that $\bigcap\limits_{n=1}^{\infty} K_n \ne \varnothing$.

Problem B.8 (6/1997)**.** Let $\{x_n\}_{n\ge 0}$ be a sequence in $\mathbb{R}^m$ with the properties that the sequences

$$\{x_{2n}\}_{n\ge0}, \quad \{x_{2n+1}\}_{n\ge0}, \quad \{x_{5n}\}_{n\ge0}$$

are convergent. Show that the sequence $\{x_n\}_{n\ge0}$ converges.

Problem B.9 (6/1997)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables and assume that the restriction of f to any line in $\mathbb{R}^2$ is differentiable, i.e. for any $a, b, c, d \in \mathbb{R}$, the function of one variable $g(t) = f(at + b, ct + d)$ is differentiable. Is f continuous on $\mathbb{R}^2$? (Either prove it or give a counterexample.)

Problem B.10 (6/1997)**.** Let u be Riemann integrable on $[a,b]$.
(a) Show that for every $\epsilon > 0$, there exists a continuous function $f : [a,b] \to \mathbb{R}$ such that

$$\int_a^b |u(x) - f(x)|\,dx < \epsilon.$$

(b) Show that there exists a sequence $\{P_n\}_{n\geq 0}$ of polynomials such that

$$\lim_{n\to\infty}\int_a^b |u(x) - P_n(x)|\, dx = 0.$$

Problem B.11 (8/1997)**.** Let $x_1 = \sqrt{2}$, $x_2 = \sqrt{2+\sqrt{2}}$ and $x_{n+1} = \sqrt{2+\sqrt{x_n}}$ if $n \geq 2$. Show that $\{x_n\}_{n=1}^{\infty}$ converges and evaluate its limit. (Justify!)

Problem B.12 (8/1997)**.** Let $f : (0,1) \to \mathbb{R}$ be a differentiable function and suppose that there exists a constant $M \geq 0$ such that

$$|f'(x)| \leq M, \quad \text{for all } x \in (0,1).$$

Show that f is uniformly continuous on $(0,1)$ and that $\lim_{x\to 0^+} f(x)$ exists.

Problem B.13 (8/1997)**.** Consider the sequence of functions $\{f_n\}_{n\geq 1}$ defined on the interval $[0,\infty)$ by

$$f_n(x) = \frac{ne^{-x} + x^2}{n+x}, \quad x \in [0,\infty),\ n \geq 1.$$

(a) Show that $\{f_n\}_{n\geq 1}$ converges uniformly on $[0,1]$ to a function f and determine f explicitly.
(b) Is $\{f_n\}$ uniformly convergent on $[0,\infty)$?
(c) Find $\lim_{n\to\infty}\int_0^1 \frac{ne^{-x}+x^2}{n+x}\, dx$. (Justify!)

Problem B.14 (12/1997)**.** Let (E,d) be a metric space and let A, B be subsets of E with A compact and B closed.
(a) Define the function $f : E \to \mathbb{R}$ by $f(x) = d(x,B) := \inf_{y\in B} d(x,y)$. Show f is continuous on E.
(b) Show that if $A \cap B = \varnothing$, then $d(A,B) > 0$, where $d(A,B) := \inf_{\substack{x\in A\\ y\in B}} d(x,y)$.

Problem B.15 (12/1997)**.** Suppose $f \in C^1(\mathbb{R})$, and $\int_{-\infty}^{\infty} |f'(x)|\, dx < \infty$. Prove that $\lim_{|n|\to\infty} f(x)$ exists

Problem B.16 (12/1997)**.** Let $\{f_n(x)\}$ be a sequence of increasing functions on $[0,1]$ (i.e., $f_n(x) \leq f_n(y)$ for all $0 \leq x \leq y \leq 1$ and all $n = 1, 2, 3, \ldots$). Suppose $\lim_{n\to\infty} f_n(x) = 0$ (pointwise). Show $f_n(x) \to 0$ uniformly for all $x \in [0,1]$ as $n \to \infty$.

Problem B.17 (5/1998)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Suppose that the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist in a neighborhood of $(0,0)$ and are continuous at $(0,0)$. Show that f is differentiable at $(0,0)$.

Problem B.18 (5/1998)**.** Compute $\lim_{n\to\infty}(n^{2/n} - 1)\dfrac{n}{\log n}$ or show that the limit does not exist.

Problem B.19 (5/1998). Let $\{f_n\}$ be a sequence of real-valued continuous functions on $[0,1]$ which converges pointwise on the interval $[0,1]$. Suppose that f_n is continuously differentiable on $(0,1)$ and that

$$\int_0^1 |f_n'(x)|^2\,dx \le 1, \quad n = 1,2,3,\ldots.$$

Prove that:
(a) $\{f_n\}_{n=1}^\infty$ is equicontinuous on $[0,1]$.
(b) $\{f_n\}_{n=1}^\infty$ converges uniformly on $[0,1]$.

Problem B.20 (8/1998). (a) Show that there exists a continuous function $f:(0,1]\to \mathbb{R}$ such that the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n\sqrt{nx+1}}$ converges uniformly to f on every interval $[\varepsilon,1]$, where $0<\varepsilon<1$.
(b) Show that the series in part (a) does not converge uniformly to f on the interval $(0,1]$ and that f is unbounded near 0.
(c) Show that the improper integral $\displaystyle\int_0^1 f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 f(x)\,dx$ exists and can be expressed as the sum of the series

$$\sum_{n=1}^\infty \frac{2(\sqrt{n+1}-1)}{n^2}. \quad \text{(Justify!)}$$

Problem B.21 (8/1998). Suppose that the function $f:(0,1)\to\mathbb{R}$ has a finite derivative at every point in $(0,1)$ and that $\lim_{x\to 1^-} f(x)=\infty$. Prove that $\lim_{x\to 1^-} f'(x)$ either fails to exist or is infinite.

Problem B.22 (8/1998). Set $a_n = 2\sqrt{n} - \displaystyle\sum_{k=1}^n \frac{1}{\sqrt{k}}$ for $n=1,2,3,\ldots$. Show that the sequence $\{a_n\}_{n\ge 1}$ is increasing and that it converges to a limit a with $1<a<2$.

Problem B.23 (4/1999).

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & \text{if } x^2+y^2>0,\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

(a) Is $f(x,y)$ continuous at $(0,0)$? Prove your assertion.
(b) Do all directional derivatives of $f(x,y)$ at $(0,0)$ exist? Prove your assertion.
(c) Is $f(x,y)$ differentiable at $(0,0)$? Prove your assertion.

Problem B.24 (4/1999). Let $f_0(x)=1$ and for $n\ge 1$ set

$$f_n(x) = f_{n-1}(x) + \frac{1}{n^2}\int_0^x \sin(f_{n-1}(t))dt, \quad 0\le x\le 1.$$

Show that $\{f_n\}$ converges uniformly on $[0,1]$. (Hint: $f_N(x) = f_0 + \sum_{n=1}^N (f_n - f_{n-1})$.)

Problem B.25 (4/1999)**.** (a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $[a,b]$ is a compact interval, then there exists $x_0 \in [a,b]$ so that $\sup_{x_0\in[a,b]} f(x) = f(x_0)$.

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable on (a,b), and f attains its maximum at $x_0 \in (a,b)$, then $f''(x_0) \leq 0$.

Problem B.26 (4/1999)**.** (a) State the Weierstrass Approximation Theorem for real valued functions on the interval $[a,b]$.
(b) A sequence $\{x_n\}$ of numbers in $[0,1]$ is called *equidistributed* if for every continuous function f on $[0,1]$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} f(x_i) = \int_0^1 f(x)\,dx.$$

Show that $\{x_n\}$ is equidistributed on $[0,1]$ if and only if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i^k = \frac{1}{k+1}, \quad \text{for all } k = 0,1,2,3,\ldots$$

Problem B.27 (8/1999)**.** Define a sequence $\{a_n\}$ by:

$$a_0 = 0, \quad a_1 = 3, \quad a_n = \tfrac{1}{2}(a_{n-1} + a_{n-2}) \quad \text{for } n > 1.$$

(a) Show that $a_n - a_{n-1} = -\frac{1}{2}(a_{n-1} - a_{n-2})$ for all $n = 2,3,4,\ldots$
(b) Use (a) to prove that $a_n - a_{n-1} = 3[-(\frac{1}{2})]^{n-1}$ for all $n = 2,3,4,\ldots$
(c) Show that $\{a_n\}$ is a Cauchy sequence.
(d) Find $\lim_{n\to\infty} a_n$.

Problem B.28 (8/1999)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous for $x \in \mathbb{R}$, differentiable for $x \in (-\infty,0) \cup (0,\infty)$, and $\lim_{x\to 0} f'(x) = 2$. Show that f is differentiable at $x = 0$ and $f'(0) = 2$.

Problem B.29 (8/1999)**.** Define a sequence $\{f_n\}$ of functions

$$f_n(x) = \frac{x}{1+nx^2}, \quad n = 1,2,\ldots.$$

(a) Show that $f_n(x)$ converges pointwise for all $x \in \mathbb{R}$. Find $f(x) = \lim_{n\to\infty} f_n(x)$.
(b) True or False: f_n converges uniformly to f on $\mathbb{R}$. Prove your assertion.
(c) True or False: $f'(x) = \lim_{n\to\infty} f_n'(x)$ holds for all $x \in \mathbb{R}$. Prove your assertion.

Problem B.30 (8/1999)**.** Suppose (x_0, y_0, u_0, v_0) is a solution to the system of equations

$$(*) \qquad \begin{cases} e^x \cos y + 2u - v = 0 \\ e^x \sin y - u + 2v = 0. \end{cases}$$

(a) Show that, in a neighborhood of (x_0, y_0, u_0, v_0), $(*)$ can be solved for (x,y) as a function of (u,v).
(I.e. all solutions of $(*)$ can be expressed in the form $(x,y) = g(u,v)$ for some C^1 function $g : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, where U is an open neighborhood of (u_0, v_0).)
(b) Show that, in a neighborhood of (x_0, y_0, u_0, v_0), $(*)$ can be solved for (u,v) as a function of (x,y).

$\big($I.e. all solutions of $(*)$ can be expressed $(u,v)=h(x,y)$ for some C^1 function $h : V \subset \mathbb{R}^2 \to \mathbb{R}^2$, where V is an open neighborhood of $(x_0,y_0)\big)$.

Problem B.31 (12/1999)**.** Suppose $\{x_n\}$ is a sequence in a complete metric space (X,d) such that

$$\sum_n d(x_n, x_{n+1})$$

is a convergent series. Show that the sequence $\{x_n\}$ is convergent in X.

Problem B.32 (12/1999)**.** (a) State the definition of uniform continuity of a function $f : X \to Y$, where X, Y are metric spaces.
(b) Explain how the Mean Value Theorem can be used to prove the uniform continuity of a function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval.
(c) Which of the intervals is the function $f(x) = \ln x$ uniformly continuous on?
  (i) the interval $(0,\infty)$ $\big($Given $\delta > 0$, can $|f(x+h) - f(x)|$ be greater than 1 for $h = \delta/2$?$\big)$
  (ii) the interval $(1,\infty)$.
Justify your answer!

Problem B.33 (12/1999)**.** (a) State the definition of a connected subset of a metric space.
(b) Prove that the continuous image of a connected set is connected.
(c) Which of the following sets is connnected as a subset of the appropriate Euclidean space?
  (i) the set $\mathbb{Q}$ of all rational numbers.
  (ii) the unit circle $\{(x,y) \in \mathbb{R}^2,\ x^2+y^2=1\}$.
  (iii) the cube without its center:

$$\{(x,y,z) \in \mathbb{R}^3,\ |x| \le 1, |y| \le 1, |z| \le 1, (x,y,z) \ne (0,0,0)\}.$$

Justify your answer!

Problem B.34 (12/1999)**.** (a) State the Stone-Weierstrass theorem.
(b) Show that the linear combinations of the powers $x^2, x^4, \ldots, x^{2n}, \ldots$ are dense in $\mathscr{C}([1,2])$, the space of continuous functions on $[1,2]$,

Problem B.35 (4/2000)**.** (a) Show that

$$\int_0^\infty \frac{\cos x}{x^{1/3}}\,dx$$

converges (as an improper integral).
(b) Show that the integral in part (a) does not converge absolutely.

Problem B.36 (4/2000)**.** (a) State the Stone-Weierstrass Theorem, explaining the required properties of the algebra in question.
(b) Show that span $\{x^j\, e^{ky} : j \ge 0,\ k \ge 0, k, j \text{ integers}\}$ is dense in $\mathscr{C}\big([0,1]\times[0,1]\big)$, the space of real-valued continuous functions on $[0,1]\times[0,1]$.
(c) Show that span $\{x^j\, e^{jy} : j \ge 0,\ j \text{ an integer}\}$ is not dense in $\mathscr{C}\big([0,1]\times[0,1]\big)$.

Problem B.37 (4/2000)**.** Let $f_n(x) = (1 + \frac{x}{n})^n$, $n \geq 1$.
(a) Show that $f_n(x)$ converges pointwise for all $x \in \mathbb{R}$ and find $f(x) = \lim_{n\to\infty} f_n(x)$.
(b) Show that $\{f_n\}$ converges uniformly on $[0, 1]$. (Hint: is the sequence $\{f_n\}$ monotone on $[0, 1]$?)
(c) Does the sequence $\{f_n\}$ converge uniformly on $[0, \infty)$? Prove your assertion.

Problem B.38 (4/2000)**.** (a) Show that the set $\{(x^2, y^3, xy) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4\}$ is compact and connected.
(b) Find all point(s) at which the mapping $T : \mathbb{R}^2 \to \mathbb{R}^3 : (x, y) \mapsto \left(x^2, y, x\,(y^2 - 1)\right)$ is not locally one-to-one and explain why T is not locally one-to-one at those point(s).

Problem B.39 (5/2001)**.** Suppose $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence of real numbers, and define

$$a_n = \inf\{x_k : \ k \geq n\}, \qquad b_n = \sup\{x_k : \ k \geq n\}.$$

(a) Show that a_n and b_n are convergent sequences, whether x_n is convergent or not.
(b) Show that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} b_n$.
(c) Show that $x_n \to L$ if and only if $\lim a_n = \lim b_n = L$.

Problem B.40 (5/2001)**.** Define $\mathscr{M} = [1, \infty)$ with distance function

$$d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|.$$

(a) Prove that $(\mathscr{M}, d)$ defines a metric space.
(b) Show that $\mathscr{M}$ is a bounded metric space but not a compact metric space.
(c) Is $\mathscr{M}$ complete? Prove your assertion.

Problem B.41 (5/2001)**.** Define a function

$$\phi_n(x) = \begin{cases} |x|, & \text{if } |x| \geq \frac{1}{n}, \\ \frac{n}{2}x^2 + \frac{1}{2n}, & \text{if } |x| < \frac{1}{n}. \end{cases}$$

(a) Show that each ϕ_n is continuous and differentiable on $\mathbf{R}$.
(b) Prove that $\phi_n \to |x|$ uniformly.
(c) Does $\phi_n'(x)$ converge uniformly? Prove your assertion.

C. Basic Complex Analysis

Problem C.1 (12/1994)**.** If $a > e$ show that the equation $e^z = az^n$ has n solutions inside the unit circle.

Problem C.2 (12/1994)**.** Let g be continuous in $[0, 1]$ and

$$F(z) = \int_0^1 \sin(zt)\; g(t)\; dt.$$

Is F analytic in the finite complex plane? If so, show it.

Problem C.3 (12/1994)**.** Evaluate (via residue calculus) the real improper integral

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\, dx,$$

where $0 < \alpha < 1$.

Problem C.4 (5/1996)**.** Describe the locus of points z given by the equation $|z-1|^2 = |z+1|^2 + 6$.

Problem C.5 (5/1996)**.** Find the radius of convergence and the set of convergence of the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n2^n}.$$

Problem C.6 (5/1996)**.** Suppose that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is entire and satisfies $|f(z)| \le Me^{|z|}$ for all $z \in \mathbb{C}$. Show that $|c_n| \le M\frac{e^n}{n^n}$ for all $n \ge 1$.

Problem C.7 (5/1996)**.** State Rouché's theorem and show that all five roots of the polynomial $P(z) = z^5 + 6z^3 + 2z + 10$ are in the annulus $\{z \in \mathbb{C},\ 1 < |z| < 5\}$.

Problem C.8 (5/1997)**.** Define $f(z) = \displaystyle\int_1^2 \frac{1}{t-z}\,dt$ for $z \in \mathbb{C} \setminus [1,2]$.
(a) Show that f is well-defined and analytic on $\mathbb{C} \setminus [1,2]$.
(b) Find a power series expansion in powers of z for f near the point $z = 0$ and determine its radius of convergence.
(c) Compute $\displaystyle\int_\gamma f(z)dz$, where $\gamma(t) = 3e^{it}$ for $0 \le t \le 2\pi$.

Problem C.9 (5/1997)**.** Consider the sequence $\{A_n\}_{n=0}^{\infty}$, where $A_0 = A_1 = 1$ and $A_{n+2} = A_{n+1} + A_n$ for $n \ge 0$. Show that

$$\sum_{n=0}^{\infty} A_n z^n = \frac{1}{1 - z - z^2}$$

whenever the series converges and find an explicit formula for the coefficients $A_n,\ n \ge 0$.

Problem C.10 (5/1997)**.** How many roots does the equation $z^7 - 5z^3 + 1 = 0$ have in the region $\{z \in \mathbb{C},\ 1 < |z| < 2\}$?

Problem C.11 (5/1997)**.** Find a one-to-one conformal mapping from the disk $D_1 = \{z \in \mathbb{C},\ |z| < 1\}$ onto $D_2 = \{w \in \mathbb{C},\ |w+1| < 1\}$ with $f(0) = -1/2$.

Problem C.12 (6/1997)**.** Let $r(z)$ be a rational function of the form $r(z) = \frac{P(z)}{Q(z)}$, where P, Q are polynomials and $\deg P < \deg Q$. Let $a_1, \ldots, a_m$ be the poles of $r(z)$ and let $S_k(z)$ be the singular part of $r(z)$ at the pole a_k, for $k = 1, \ldots, m$. Show that $r(z) = \sum_{i=1}^{m} S_k(z)$.

Problem C.13 (6/1997)**.** Let $D = \{z \in \mathbb{C},\ |z| < 1\}$ and let

$$\mathscr{F} = \left\{ f : D \to \mathbb{C}, f(z) = \sum_{n=0}^{\infty} a_n z^n, |a_n| \le 1, z \in D \right\}.$$

Show that each $f \in \mathscr{F}$ is well-defined and analytic on D and that $\mathscr{F}$ is a normal family.

Problem C.14 (6/1997)**.** Suppose that f is analytic on a region $\Omega \subset \mathbb{C}$ and that $f(z) \neq 0$ for all $z \in \Omega$. Let $z_0 \in \Omega$ and assume that there exists a neighborhood V of z_0 in Ω such that

$$|f(z_0)| \leq |f(z)|, \quad \text{for all } z \in V.$$

Prove that f is constant in Ω.

Problem C.15 (6/1997)**.** Let $f(z) = \dfrac{z^2 - z - 1}{(z-1)^2(z-2)}$. Find the Laurent series expansion of $f(z)$ in the annulus $\{z \in \mathbb{C},\ 1 < |z| < 2\}$.

Problem C.16 (8/1997)**.** Use the calculus of residues to compute the integral

$$\int_{-\infty}^{\infty} \frac{2x^2 - 1}{x^4 + 5x^2 + 4}\, dx.$$

Problem C.17 (8/1997)**.** Find a one-to-one conformal mapping from $D_1 = \{z \in \mathbb{C},\ |z| < 1\}$ onto $D_2 = \{w \in \mathbb{C},\ |w - 1| < 1\}$ with $f(0) = 1/2$ and $f'(0) > 0$.

Problem C.18 (8/1997)**.** Let f and g be analytic on $D = \{z \in \mathbb{C},\ |z| < 1\}$ and continuous on $\overline{D}$. Suppose that $\mathrm{Im}(f(0) = \mathrm{Im}(g(0))$ and $\mathrm{Re}(f(z)) = \mathrm{Re}(g(z))$ for all $z \in \mathbb{C}$ with $|z| = 1$. Show that $f(z) = g(z)$ for all $z \in \mathbb{C}$ with $|z| \leq 1$.

Problem C.19 (12/1997)**.** Let f be an analytic in the punctured disk $0 < |z| < 1$ and suppose that $\mathrm{Re}(f(z)) < 0$ for all $0 < |z| < 1$. Show that $z = 0$ is a removable singularity of f.

Problem C.20 (12/1997)**.** Suppose T is a one-to-one conformal mapping from the unit disk $D = \{z \in \mathbb{C},\ |z| < 1\}$ to itself. Prove that there exist $\theta \in \mathbb{R}$ and $z_0 \in D$ such that $T(z) = e^{i\theta} \frac{z - z_0}{1 - z z_0}$ for all $z \in D$.

Problem C.21 (12/1997)**.** Let $U = \mathbb{C} \setminus \{z \in \mathbb{R},\ z \leq 0\}$. If $\log z$ denotes the principal branch of the logarithm defined on U, let $z^{1/2}$ be defined by $z^{1/2} = e^{1/2 \log z}$ for $z \in U$. Compute $\int_\gamma z^{1/2} dz$ if $\gamma(t) = 2e^{it}$ for $-\pi/2 \leq t \leq \pi/2$.

Problem C.22 (8/1998)**.** Let $R(z)$ be a rational function of the form $R(z) = \frac{P(z)}{Q(z)}$, where P and Q are two polynomials with no common factor. Suppose that $|R(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$. Show that there exist an integer $r \in \mathbb{Z}$ and complex constants C and $\rho_1, \ldots, \rho_m$, with $|C| = 1,\ 0 \neq |\rho_i| \neq 1,\ i = 1, \ldots, m$ such that

$$R(z) = C z^r \prod_{i=1}^{m} \frac{z - \rho_i}{1 - \overline{\rho}_i z}.$$

Problem C.23 (8/1998)**.** Let $f(z) = \displaystyle\sum_{n=1}^{\infty} a_n z^n,\ |z| < 1$, where the coefficients a_n satisfy $|a_n - n| \leq 2^{-n}$ for all $n \geq 0$.

(a) Show that f is well-defined and analytic on the open disk $\{z \in \mathbb{C}, |z| < 1\}$.

(b) Show that f can be extended analytically on the open set $\{z \in \mathbb{C}, |z| < 2\} \setminus \{1\}$ and compute the integral

$$\int_{\mathfrak{A}} f(z)dz,$$

where $\mathfrak{A}$ denotes the circle centered at 0 with radius 3/2 described in the counterclockwise direction.

Problem C.24 (8/1998)**.** Find all the one-to-one conformal mappings φ from the unit disk $D = \{z \in \mathbb{C}, |z| < 1\}$ to the upper-half plane $H^+ = \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$ satisfying $\varphi(1/2) = i$.

Problem C.25 (4/1999)**.** Define

$$f(z) = \int_{-1}^{1} \frac{dt}{t - z}.$$

Here the integral is over the real interval $t \in [-1, 1]$.
(a) Show that f defines an analytic function for

$$z \in \mathbb{C} \setminus \{z = x + i0 \mid x \in [-1, 1]\}.$$

(b) Compute $\displaystyle\int_{|z|=3} f(z)dz$.

Problem C.26 (4/1999)**.** Answer **True** or **False**. If "true" give a brief justification. If "false" give a counterexample. In each part, assume that $f(z)$ is **entire**.
(a) If $\operatorname{Re} f(z) > 10$ for all complex z, then $f(z)$ is constant.
(b) If $f(z) = 0$ for all $z = x + i0,\ x \in [-1, 1]$, then $f(z) \equiv 0$.
(c) If $|f(z)| > \frac{1}{10}$ for all complex z, then $f(z)$ is constant.
(d) If $|f(z)|$ is bounded on the real line $z = x + i0$, then $f(z)$ is constant.

Problem C.27 (4/1999)**.** Evaluate $\displaystyle\int_0^\infty \frac{x^{-1/2}}{1 + x^2}\, dx$ using contour integration.

Problem C.28 (8/1999)**.** (a) Find the radius of convergence of the Maclaurin series for the function $f(z) = 1/(e^z - 1)$.
(b) Compute the first four non-zero terms of the series in part (a).

Problem C.29 (8/1999)**.** Use the calculus of residues to evaluate the integral

$$\int_0^{2\pi} \frac{1}{5 + 3\cos(\theta)}\, d\theta.$$

Problem C.30 (8/1999)**.** Compute the Laurent series expansion of the function $f(z) = \dfrac{3z + 5}{z^2 + 3z + 2}$ on the following annuli:

(a) $1 < |z| < 2$
(b) $|z| > 2$

Problem C.31 (4/2000)**.** Let $f(z), g(z)$ be two functions analytic in some connected open set $U \subset \mathbb{C}$ with g being not identically zero. Show that

$$f(z)\,\overline{g(z)} \geq 0, \quad \forall\, z \in \mathbb{C}$$

if and only if $f(z) = C g(z)$ for some constant $C \geq 0$.

Problem C.32 (4/2000)**.** (a) Let $f(z)$ be analytic on the disk $D_R = \{z,\, |z| < R\}$, where $R > 0$, and suppose that $|f(z)| \leq M$, for all $z \in D_R$. Show that

$$|f^{(n)}(z)| \leq \frac{M\,R\,n!}{(R - |z|)^{n+1}}, \quad \forall\, z \in D_R, \forall\, n \geq 0.$$

(b) Suppose that $g(z) = \sum_{n=0}^{\infty} c_n\, z^n$ is entire and satisfies

$$|g(z)| \leq M\, e^{|z|^2}, \quad \forall\, z \in \mathbb{C}.$$

Show that

$$|c_n| \leq M\, \left(\frac{2\,e}{n}\right)^{n/2}, \quad \forall\, n \geq 1.$$

Problem C.33 (4/2000)**.** Compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x - \pi/4)^2 + a^2}\, dx,$$

where $a > 0$, using the calculus of residues.

Problem C.34 (4/2000)**.** Let $P(z) = z^n + a_{n-1}\, z^{n-1} + \cdots + a_1\, z + a_0$, where $a_0, \ldots, a_{n-1}$ are complex numbers. Show that either $|P(z)| > 1$ for some z with $|z| = 1$ or $P(z) = z^n$.

Problem C.35 (8/2000)**.** Use contour integration to evaluate

$$\int_0^{\infty} \frac{\ln x}{4 + x^2}\, dx.$$

Problem C.36 (8/2000)**.** What type of singularity does

$$f(z) = \frac{1}{e^{iz^2} - 1}$$

have at $z_0 = 0$? Find the first 4 terms of its Laurent expansion about $z_0 = 0$.

Problem C.37 (8/2000)**.** Suppose f is entire and $|f(z)| \to \infty$ as $|z| \to \infty$. Show that $f(z)$ is a polynomial.

Problem C.38 (8/2000)**.** (a) Suppose f is analytic in a domain D, and $|f(z)| \leq 10$ for all $z \in D$. Show that

$$|f'(z)| \leq \frac{10}{d(z)},$$

where $d(z)$ is the distance from z to the boundary of D. (Hint: use the Cauchy Integral Formula.)
(b) State and prove Liouville's theorem for bounded entire functions.

Problem C.39 (5/2001)**.** Define $f(z)$ by the series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots$$

(a) Show that $f(z)$ is analytic for $|z| < 1$, but $f(z)$ is singular at $z = 1$.
(b) Show that $f(z^2) = f(z) - z$ and conclude that $f(z)$ is singular at $z^2 = 1$.
(c) Show that $f(z)$ is singular at the m points $z^m = 1$ on the unit circle for any $m = 1, 2, 2^2, 2^3, \cdots$. Conclude that $f(z)$ is singular on the entire unit circle $|z| = 1$.

Problem C.40 (5/2001)**.** Suppose that $f(z)$ is a meromorphic function on the unit disk D (i.e., its most severe singularities are isolated poles). Its zeros within D occur at points p_j, $j = 1, \ldots J$, of degrees m_j respectively, and its poles occur at points q_k, $k = 1, \ldots K$, with orders n_k respectively.
(a) Prove that

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{J} m_j - \sum_{k=1}^{K} n_k.$$

(b) Compute the following contour integral:

$$\oint_{|z|=3} \frac{2z^3 - 3z}{z^4 - 3z^2 - 4} dz$$

Problem C.41 (5/2001)**.** Suppose S, T are domains in $\mathbb{C}$, $f(z) : S \to T$ a conformal mapping, and $h(\zeta) = h(u + iv)$ a harmonic function defined on T.
(a) Show that the function $g(x, y) = h \circ f(x + iy)$ is a harmonic function on S.
(b) Take

$$f(z) = \mathrm{Log}\,(z) := \ln r + \, i\,\theta, \qquad -\pi < \theta < \pi,$$

and consider the harmonic function $h(u, v) = u + 2v$ on the region

$$T = \{\zeta = u + iv : u > 0, \ \frac{\pi}{4} < v < \frac{\pi}{2}\}.$$

What is the preimage S of T under this map?
What is the corresponding harmonic function $g(x, y)$ on S?