PROBABILITY AND STATISTICS PRELIMINARY EXAM

Please answer four questions on part A and three questions on part B. All questions are weighted evenly. Please provide clear and complete explanations of all steps taken, and make sure to justify any assumptions you make in the process. Good luck!

A. Core material

Answer four of the following six questions.

Problem A.1. Let V be an inner product space over $\mathbb{R}$ and $T : V \to V$ an orthogonal linear transformation.

(a) Show that every eigenvalue of T has absolute value 1.
(b) Suppose that W is a T-invariant subspace of V. Show that the orthogonal complement of W is T-invariant.

Problem A.2. Find an explicit formula for the entries of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$ in term of n

Problem A.3. Let f be a real-valued *bounded* monotonic function on the interval $(0, 1)$. Show that if f is continuous on $(0, 1)$, then it is also uniformly continuous there.

Problem A.4. Let K be a compact metric space and let $\{f_n\}$ be a sequence of real-valued continuous functions on K that converges uniformly to a function f on K. Show that:

(a) f is continuous on K.
(b) $\{f_n\}$ is bounded and equicontinuous on K.

Problem A.5. Suppose f is an entire function such that $|f(z)| \leq |\exp(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant C such that $f(z) = C \exp(z)$ for all $z \in \mathbb{C}$.

Problem A.6. Let a be a positive real number. Use the Calculus of Residues to show that $\displaystyle\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx = \frac{\pi}{4a}$.

Date: 9:30–12:30 on Dec 21, 2006 in HH/410.

B. Probability and Statistics

Answer three of the following four questions.

Problem B.1. (1) Suppose that the number of colds that a person contracts in a given year is a Poisson random variable with $\lambda = 5$. A new drug based on large quantities of Vitamin C is being marketed. This drug is effective for 75% of the population in that it decreases the Poisson parameter to $\lambda = 3$ for these people. For the other 25% of the population the drug has no appreciable effect on colds. If an individual tries the drug for one year and has 3 colds. What is the probability that the drug is effective for this person?

(2) Suppose that X is a beta random variable with parameters α and β.

(a) Show that the r^{th} moment of the distribution of X is

$$\mathrm{E}[X^r] = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+r)}{\Gamma(\alpha+\beta+r)\Gamma(\alpha)}$$

where $\Gamma(a)$ is the gamma function evaluated at a.

(b) Use the above result to derive the mean and variance of X. (**Hint:** Recall $\Gamma(a+1) = a\Gamma(a)$).

Problem B.2. Suppose that X and Y are jointly continuous random variables with joint probability density function

$$f(x,y) = kxy \qquad 0 < x < y < 1$$

(1) Find the value of k which makes this a valid joint probability density function.
(2) Calculate $\Pr(2X > Y)$.
(3) Find the conditional density function of X given that $Y = y$ and hence show whether or not X and Y are independent.
(4) Calculate $E[(Y-X)^2]$.

Problem B.3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability mass function

$$f(x) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

(1) Show that the mle of θ is given by $\widehat{\theta} = -\dfrac{n}{\sum_{i=1}^{n} \log X_i}$.
(2) Show that $W = -\sum_{i=1}^{n} \log X_i$ has the gamma distribution $\Gamma(n, 1/\theta)$.
(3) Show that $2\theta W$ has a $\chi^2(2n)$ distribution.
(4) Use the result of (c) to derive a $100(1-\alpha)\%$ confidence interval for θ.

Problem B.4. Let $\overline{X}_n$ denote the mean of a random sample of size n from a Poisson distribution with parameter $\mu = 1$.

(1) Show that the mgf of $Y_n = \sqrt{n}(\overline{X}_n - 1)$ is given by $\exp[-t\sqrt{n} + n(e^{t/\sqrt{n}} - 1)]$.
(2) Use part (a) to find the limiting distribution of Y_n as $n \to \infty$. (**Hint:** Replace, by its MacLaurin series, the expression $e^{t/\sqrt{n}}$, which is in the exponent of the mgf of Y_n).

Some Distributions

Distribution	mgf
Binomial $p(x) = \binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$[p + (1-p)e^t]^n$
Poisson $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, \ldots$	$e^{\lambda(e^t - 1)}$
Exponential $f(x) = \frac{1}{\beta} e^{-x/\beta}$, $x > 0$	$\frac{1}{1-\beta t}$
Double Exponential $f(x) = \frac{1}{2} e^{-\lvert x \rvert}$, $-\infty < x < \infty$	$\frac{1}{1-t^2}$
Uniform(a, b) $f(x) = \frac{1}{b-a}$, $a < x < b$	$\frac{(e^{tb} - e^{ta})}{t(b-a)}$
Gamma $f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$, $0 < x < \infty$	$\frac{1}{(1-\beta t)^\alpha},\ t < \frac{1}{\beta}$
Chi-Square $f(x) = \frac{1}{\Gamma(r/2) 2^{r/2}} x^{r/2-1} e^{-x/2}$, $0 < x < \infty$	$(1-2t)^{-r/2},\ t < \frac{1}{2}$
Beta Distribution $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$, $0 < x < 1$	
Normal $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$, $-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$
t-distribution $g(t) = \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\Gamma(r/2)} \frac{1}{(1+t^2/r)^{(r+1)/2}}$, $-\infty < t < \infty$	
F-Distribution $g(w) = \frac{\Gamma[(r_1+r_2)/2](r_1/r_2)^{r_1/2}}{\Gamma(r_1/2)\Gamma(r_2/2)} \frac{w^{r_1/2-1}}{(1+r_1 w/r_2)^{(r_1+r_2)/2}}$, $0 < w < \infty$	