PROBABILITY & STATISTICS PRELIMINARY EXAM

Please answer four questions on part A and three questions on part B. All questions are weighted evenly. Please provide clear and complete explanations of all steps taken, and make sure to justify any assumptions you make in the process. Good luck!

A. Core material

Answer four of the following six questions.

Problem A.1. Let

$$A = \begin{bmatrix} 8 & 6 & -20 \\ -3 & -1 & 16 \\ 0 & 0 & 4 \end{bmatrix}$$

Consider the inner product on $\mathbb{R}^3$ given by

$$\langle v, w \rangle = v^T A w,$$

where $v, w \in \mathbb{R}^3$ are viewed as column vectors and the superscript T denotes the transpose. Find a basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ and numbers $c_1, c_2, c_3 \in \mathbb{R}^3$ such that if $v = a_1 v_1 + a_2 v_2 + a_3 v_3$ and $w = b_1 v_1 + b_2 v_2 + b_3 v_3$ for some $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}$, then

$$\langle v, w \rangle = a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3.$$

Problem A.2. Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

(a) Determine the Jordan canonical form of M.
(b) For each positive integer n, determine the Jordan canonical form of M^n.

Problem A.3. Consider the mapping $\mathbf{f} = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{f}(x, y) = (x + y, 2\,x\,y), \quad (x, y) \in \mathbb{R}^2.$$

Show that the mapping $\mathbf{f}$ is locally invertible at the point $(x_0, y_0) = (2, -1)$. Find an explicit formula for its local inverse $\mathbf{g}(u, v)$ defined in a neighborhood of $\mathbf{f}(2, -1) = (1, -4)$ and compute $\mathbf{g}'(1, -4)$.

Problem A.4. Given a metric space (X, d) and a subset $A \subset X$ with $A \neq \emptyset$, define

$$\rho(x) = \inf_{z \in A} d(x, z).$$

Show that

$$|\rho(x) - \rho(y)| \leq d(x, y), \quad \text{for all } x, y \in X.$$

Problem A.5. Use the Calculus of Residues to evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 1} dx$.

Problem A.6. Let $B = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the closed unit disk. Suppose $f : B \to \mathbb{C}$ is a continuous function which is analytic in the interior of B. Furthermore, suppose $f(0) = \frac{1}{4}$ and $|f(\exp(it))| = \frac{5}{4} - \cos(t)$ for $0 \leq t \leq 2\pi$. Show that f has a zero in the interior of B.

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B. Probability & Statistics

Answer three of the following four questions.

Problem B.1. (a) Batteries sold under a no-name brand can come from one of two factories. 70% of the batteries sold are produced in Factory A and have a lifetime (in hours) which is exponentially distributed with rate $\lambda = 0.01$. The rest of the batteries are produced in Factory B and have a lifetime (in hours) which is exponentially distributed with rate $\lambda = 0.02$.

(i) What is the probability that a randomly chosen battery will last longer than 80 hours?

(ii) If the chosen battery does in fact last longer than 80 hours, what is the probability that it was produced in Factory A?

(b) Suppose that X has a negative binomial distribution with pmf

$$f(x; r, \theta) = \binom{x+r-1}{r-1} \theta^r (1-\theta)^x, \quad x = 0, 1, 2, ..., \quad r = 1, 2, ..., \quad 0 < \theta < 1.$$

(i) Derive the moment generating function of X.

(ii) Using the result in (i), prove in detail that the sum of independent geometric random variables with common probability of success has a negative binomial distribution.

Problem B.2. Suppose that X and Y are jointly continuous random variables with pdf

$$f(x, y) = k(x-y)^2, \qquad 0 < x < 1, \quad 0 < y < 1.$$

(a) Find the value of k which makes this a valid joint probability density function.

(b) Calculate $\Pr(X < 2Y)$.

(c) Calculate the covariance of X and Y.

Problem B.3. Let $X_1, X_2, ..., X_n$ be a random sample from a population with pdf

$$f(x; \theta) = \frac{3\theta^3}{x^4}, \quad \theta < x < \infty, \quad \theta > 0.$$

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics.

(a) Show that $X_{1:n}$ is a sufficient statistic for θ.

(b) If $\widehat{\theta} = aX_{1:n}$ has to be an unbiased estimator of θ, what should be a?

(c) Derive the maximum likelihood estimator of θ and compare it to the unbiased estimator from part (b) in terms of the mean square error.

Problem B.4. Suppose that X_1, X_2, X_3 is a random sample from Gamma(3,θ) distribution with pdf

$$f(x; \theta) = \frac{1}{2\theta^3} \mathrm{e}^{-x/\theta} x^2, \quad 0 < x < \infty, \quad \theta > 0.$$

Suppose that we are interested in testing

$$\mathrm{H}_0: \ \theta = 10 \qquad \text{vs.} \qquad \mathrm{H}_1: \ \theta \neq 10.$$

(a) Derive the likelihood ratio test procedure.

(b) Determine the exact critical region if the level of significance is chosen to be 5%.