PURE MATH PRELIMINARY EXAM

Please answer four questions on part A and three questions on part B. All questions are weighted evenly. Please provide clear and complete explanations of all steps taken, and make sure to justify any assumptions you make in the process. Good luck!

A. Core material

Answer four of the following six questions.

Problem A.1. Let V be an inner product space over $\mathbb{R}$ and $T : V \to V$ an orthogonal linear transformation.

(a) Show that every eigenvalue of T has absolute value 1.
(b) Suppose that W is a T-invariant subspace of V. Show that the orthogonal complement of W is T-invariant.

Problem A.2. Find an explicit formula for the entries of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$ in term of n

Problem A.3. Let f be a real-valued *bounded* monotonic function on the interval $(0, 1)$. Show that if f is continuous on $(0, 1)$, then it is also uniformly continuous there.

Problem A.4. Let K be a compact metric space and let $\{f_n\}$ be a sequence of real-valued continuous functions on K that converges uniformly to a function f on K. Show that:

(a) f is continuous on K.
(b) $\{f_n\}$ is bounded and equicontinuous on K.

Problem A.5. Suppose f is an entire function such that $|f(z)| \leq |\exp(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant C such that $f(z) = C\exp(z)$ for all $z \in \mathbb{C}$.

Problem A.6. Let a be a positive real number. Use the Calculus of Residues to show that $\displaystyle\int_0^\infty \frac{x^2}{(x^2 + a^2)^2} dx = \frac{\pi}{4a}$.

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B. Pure math

Answer three of the following four questions.

Problem B.1. Let $p < q$ be two primes and G a group of order pq^3. Show that G has a normal subgroup.

Problem B.2. Let G be a group, and let H and K be normal subgroups with $H \cap K = \{e\}$. Show that $hk = kh$ for all $h \in H$ and all $k \in K$.

Problem B.3. Let α be a root of the polynomial $x^5 + 1$ different from -1. Show that $[\mathbb{Q}[\alpha] : \mathbb{Q}] = 4$.

Problem B.4. Let R be a unique factorization domain.

(a) Prove that if P is an nonzero principal ideal of R then P must contain an irreducible element.

(b) A prime ideal P in an integral domain R is called *minimal* if $P \neq 0$ and the only prime ideals Q such that $Q \subseteq P$ are $Q = 0$ and $Q = P$. Prove that in a unique factorization domain every minimal prime is principal.