PURE MATH PRELIMINARY EXAM

Please answer four questions on part A and three questions on part B. All questions are weighted evenly. Please provide clear and complete explanations of all steps taken, and make sure to justify any assumptions you make in the process. Good luck!

A. Core material

Answer four of the following six questions.

Problem A.1. Let

$$A = \begin{bmatrix} 8 & 6 & -20 \\ -3 & -1 & 16 \\ 0 & 0 & 4 \end{bmatrix}$$

Consider the inner product on $\mathbb{R}^3$ given by

$$\langle v, w \rangle = v^T A w,$$

where $v, w \in \mathbb{R}^3$ are viewed as column vectors and the superscript T denotes the transpose. Find a basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ and numbers $c_1, c_2, c_3 \in \mathbb{R}^3$ such that if $v = a_1v_1 + a_2v_2 + a_3v_3$ and $w = b_1v_1 + b_2v_2 + b_3v_3$ for some $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}$, then

$$\langle v, w \rangle = a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3.$$

Problem A.2. Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

(a) Determine the Jordan canonical form of M.
(b) For each positive integer n, determine the Jordan canonical form of M^n.

Problem A.3. Consider the mapping $\mathbf{f} = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{f}(x, y) = (x + y, 2\,x\,y), \quad (x, y) \in \mathbb{R}^2.$$

Show that the mapping $\mathbf{f}$ is locally invertible at the point $(x_0, y_0) = (2, -1)$. Find an explicit formula for its local inverse $\mathbf{g}(u, v)$ defined in a neighborhood of $\mathbf{f}(2, -1) = (1, -4)$ and compute $\mathbf{g}'(1, -4)$.

Problem A.4. Given a metric space (X, d) and a subset $A \subset X$ with $A \neq \emptyset$, define

$$\rho(x) = \inf_{z \in A} d(x, z).$$

Show that

$$|\rho(x) - \rho(y)| \le d(x, y), \quad \text{for all } x, y \in X.$$

Problem A.5. Use the Calculus of Residues to evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 1} dx.$

Problem A.6. Let $B = \{z \in \mathbb{C} \mid |z| \le 1\}$ be the closed unit disk. Suppose $f : B \to \mathbb{C}$ is a continuous function which is analytic in the interior of B. Furthermore, suppose $f(0) = \frac{1}{4}$ and $|f(\exp(it))| = \frac{5}{4} - \cos(t)$ for $0 \le t \le 2\pi$. Show that f has a zero in the interior of B.

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B. Pure math

Answer three of the following four questions.

Problem B.1. There are precisely two groups up to isomorphism of order pq, where p and q are both primes and $p|(q-1)$. Describe them.

Problem B.2. Let G be an infinite simple group. Prove that G has no non-trivial subgroups of finite index. Hint: Suppose G did have a subgroup H of finite index n. Consider the action of G on the cosets of H.

Problem B.3. Suppose R is a Principal Ideal Domain. Show that a nonzero ideal $A \subset R$ is prime if and only if it is maximal.

Problem B.4. Let $f(x) = x^5 - x - 1 \in \mathbb{Q}[x]$.

(a) Show that $f(x)$ is irreducible.

(b) Part (a), together with basic results, tells us that the quotient ring $\mathbb{Q}[x]/\langle f(x)\rangle$ is a field. Assuming that is the case, find the inverse to $g(x) = x^2 + x + 1$ in $\mathbb{Q}[x]/\langle f(x)\rangle$.