PURE MATH PRELIMINARY EXAM

Please answer four questions on part A and three questions on part B. All questions are weighted evenly. Please provide clear and complete explanations of all steps taken, and make sure to justify any assumptions you make in the process. Good luck!

A. Core material

Answer four of the following six questions.

Problem A.1. Consider the matrix $A \in M_{2\times 2}(\mathbb{R})$ given by $A = \begin{bmatrix} 5 & -6 \\ 4 & -5 \end{bmatrix}$.

(a) Show that A is diagonalizable over $\mathbb{Z}$.
(b) Find all matrices $X \in M_{2\times 2}(\mathbb{R})$ that satisfy $X^2 = A$. How many distinct square roots does A have?

Problem A.2. For $x, y \in \mathbb{R}$, let $M = M(x, y)$ be the 3×3 matrix

$$M(x,y) = \begin{bmatrix} x & y & y \\ x & x & y \\ x & x & x \end{bmatrix}.$$

Show that $\operatorname{rank}(M)$ takes on all possible values, and determine the sets of (x, y) such that

(a) $\operatorname{rank} M(x,y) = 3$.
(b) $\operatorname{rank} M(x,y) = 2$.
(c) $\operatorname{rank} M(x,y) = 1$.
(d) $\operatorname{rank} M(x,y) = 0$.

Problem A.3. (a) State the Weierstrass approximation theorem.
(b) Let $\varepsilon > 0$ and suppose $f : [0,1] \to \mathbb{R}$ is a differentiable function whose derivative f' is continuous on $[0,1]$ (i.e. f is C^1 on $[0,1]$). Prove that there exists a polynomial p such that $|f(x) - p(x)| \le \varepsilon$ and $|f'(x) - p'(x)| \le \varepsilon$ for all $x \in [0,1]$.

Problem A.4. Suppose f and f' are continuous on $[0, \infty]$ and $f(x) = 0$ for $x \ge 10^{10}$. Show that

$$\int_0^\infty f(x)^2 dx \le 2\sqrt{\int_0^\infty x^2 f(x)^2 dx}\sqrt{\int_0^\infty f'(x)^2 dx}.$$

Problem A.5. (a) Give a precise statement of the fundamental theorem of algebra.
(b) Prove the fundamental theorem of algebra.

Problem A.6. Let a be a complex number with $|a| > 1$. Evaluate the path integral around the unit circle in $\mathbb{C}$:

$$\int_{|z|=1} \frac{|dz|}{|az-1|^2}.$$

(Note that $|dz|$ represents integration with respect to arc-length.)

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B. Pure math

Answer three of the following four questions.

Problem B.1. Let G be a group and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1 g_2 g_3 = e = h_1 h_2 h_3$. Show that there exists an $a \in G$ such that the assignment $\psi(x) = a\phi(x)$ defines a homomorphism $\psi : G \to G$.

Problem B.2. Classify finite groups of order 28 up to isomorphism. How many of them are there? How many of them are nonabelian?

Problem B.3. Let $f(x) = x^4 - 2x^3 - 5x + 2$ and $g(x) = x^5 + 5x^2 - 8x^3 - 6x + 2$. Show that there is an integer d such that the polynomials $f(x)$ and $g(x)$ have a common root in $\mathbb{Q}(\sqrt{d})$. What is d?

Problem B.4. Let $\mathscr{I}$ be the ideal $\mathscr{I} = \langle x^3 + x^2 + 1, 7 \rangle$ in the ring $\mathbb{Z}[x]$. Is $\mathscr{I}$ a prime ideal of $\mathbb{Z}[x]$? Why or why not?