SAMPLE APPLIED MATH PRELIMINARY EXAM

A. Core Material

Answer four of the following six questions.

Problem A.1. Consider the matrix $A = \begin{bmatrix} -28 & 18 \\ -54 & 35 \end{bmatrix}$.

(a) Find a real matrix B such that $B^3 = A$.
(b) How many distinct complex matrices X satisfy $X^3 = A$?

Problem A.2. Let n be a positive integer and V_n the complex vector space of $n \times n$ complex matrices. For $A, B \in V_n$, define $\langle A, B \rangle = \operatorname{tr}(AB^*)$, where B^* denotes the conjugate transpose of B and $\operatorname{tr}(X)$ denotes the trace of a matrix X.

(a) Show that $\langle \cdot, \cdot \rangle : V_n \times V_n \to \mathbb{C}$ as above defines an inner product on V_n.
(b) Find the orthogonal complement of the subspace of diagonal matrices.

Problem A.3. Suppose $\{x_n\}$ is a sequence in a complete metric space (X, d) such that

$$\sum_n d(x_n, x_{n+1})$$

is a convergent series. Show that the sequence $\{x_n\}$ is convergent in X.

Problem A.4. (a) Show that

$$\int_0^\infty \frac{\cos x}{x^{1/3}}\, dx$$

converges (as an improper integral).
(b) Show that the integral in part (a) does not converge absolutely.

Problem A.5. Find all the one-to-one conformal mappings φ from the unit disk $D = \{z \in \mathbb{C}, |z| < 1\}$ to the upper-half plane $H^+ = \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$ satisfying $\varphi(1/2) = i$.

Problem A.6. (a) Suppose f is analytic in a domain D, and $|f(z)| \le 10$ for all $z \in D$. Show that

$$|f'(z)| \le \frac{10}{d(z)},$$

where $d(z)$ is the distance from z to the boundary of D. (Hint: use the Cauchy Integral Formula.)
(b) State and prove Liouville's theorem for bounded entire functions.

Date: Version 2, Sep. 11, 2005.

B. Applied Math Topics

Answer three of the following four questions.

Problem B.1. Consider the autonomous system:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + xy \\ y - x^2 \end{pmatrix},$$

Find all equilibrium points and determine their stability as $t \to +\infty$ and as $t \to -\infty$. Draw a phase portrait in the (x, y)-plane.

Problem B.2. (a) Define e^{At}, where A is an $n \times n$ matrix and $t \in \mathbb{R}$.

(b) Consider the matrix

$$A = \begin{bmatrix} -2 & 3 & 5 \\ 0 & -2 & 4 \\ 0 & 0 & -2 \end{bmatrix}.$$

(i) By writing A as the sum of a diagonal matrix and an nilpotent matrix, find e^{At}.

(ii) Is the equilibrium for the system $x' = Ax$, stable, asymptotically stable, or unstable?

(iii) Solve the initial value problem $x' = Ax + b$ where A is the given matrix, $b = [2, 0, 0]^T$, and $x(3) = (2, 1, 0)$.

(iv) Find $\lim_{t \longrightarrow \infty} x(t)$ where $x(t)$ is your solution to (iii).

Problem B.3. Use the energy method to prove the uniqueness of solutions of the heat equation with Neumann boundary conditions:

$$u_t = u_{xx} + f(x; t); \quad x \in [0; L], \ t > 0$$
$$u_x(0; t) = h_0(t); \quad u_x(L; t) = h_L(t)$$
$$u(x; 0) = \phi(x).$$

Problem B.4. Solve the nonhomogeneous wave equation:

$$\begin{aligned} u_{tt} &= u_{xx} + g(x)\sin\omega t; \quad x \in [0; \pi]; \ t > 0 \\ u(0; t) &= 0 = u(\pi; t) \\ u(x; 0) &= 0 = u_t(x; 0), \end{aligned}$$

where $g(x)$ is a continuous function for $x \in [0; L]$ such that $g(0) = g(L) = 0$. For which values of ω does the solution become unbounded in t (i.e. resonance)?