SAMPLE APPLIED MATH PRELIMINARY EXAM

A. Core Material

Answer four of the following six questions.

Problem A.1. Let V be a real vector space with a positive definite inner product $\langle\ ,\ \rangle$, and let $v_1, \cdots, v_n$ be vectors in V. Show that the $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} = \langle v_i, v_j \rangle$ is invertible if and only if the vectors $v_1, \cdots, v_n$ are linearly independent.

Problem A.2. Let V be the vector space of 2×2 matrices over the reals and suppose $A = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix}$. Define the linear transformation $T : V \to V$ by $T(X) = AXB$. Calculate the trace and the determinant of T.

Problem A.3. (a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $[a, b]$ is a compact interval, then there exists $x_0 \in [a, b]$ so that $\sup_{x \in [a,b]} f(x) = f(x_0)$.

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable on (a, b), and f attains its maximum at $x_0 \in (a, b)$, then $f''(x_0) \leq 0$.

Problem A.4. Define $\mathscr{M} = [1, \infty)$ with distance function

$$d(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$

(a) Prove that $(\mathscr{M}, d)$ defines a metric space.
(b) Show that $\mathscr{M}$ is a bounded metric space but not a compact metric space.
(c) Is $\mathscr{M}$ complete? Prove your assertion.

Problem A.5. Find a one-to-one conformal mapping from the disk $D_1 = \{z \in \mathbb{C},\ |z| < 1\}$ onto $D_2 = \{w \in \mathbb{C},\ |w + 1| < 1\}$ with $f(0) = -1/2$.

Problem A.6. Use contour integration to evaluate

$$\int_0^\infty \frac{\ln x}{4 + x^2}\, dx.$$

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B. Applied Math Topics

Answer three of the following four questions.

Problem B.1. (a) State the Poincaré–Bendixson theorem.

(b) Use this theorem, and the function $V(x,y) = x^2 + y^2$, to prove that the system

$$\begin{cases} \dot{x} = y - x^3 + x \\ \dot{y} = -x - y^3 + y \end{cases}$$

has at least one closed path (periodic orbit) in the phase plane.

Problem B.2. Consider the system

$$\begin{cases} \dot{x} = y(x+1) \\ \dot{y} = x(1+y^3) \end{cases}$$

(a) Find all equilibrium points and determine their stability as $t \to \infty$ and $t \to -\infty$.

(b) Find the 0-isoclines and ∞-isoclines and determine the time direction on these curves.

(c) Sketch a detailed phase portrait.

Problem B.3. Consider the heat equation with variable dissipation:

$$u_t = u_{xx} - 3t^2 u$$

with the initial condition $u(x;0) = \phi(x)$ for $x \in \mathbf{R}$. Find a change of variables $u(x;t) = w(t)v(x;t)$ and reduce the equation to the heat equation for $v(x;t)$. Find a solution of the problem for $u(x;t)$.

Problem B.4. (a) Give a simple complex analysis argument to show that the two-dimensional Laplace equation in plane polar coordinates (r,θ) has solutions of the form

$$\sum_n a_n r^n \cos n\theta + \sum_n b_n r^n \sin n\theta,$$

where n denotes a positive integer. Find the function $u(r,\theta)$ that satisfies the two-dimensional Laplace equation in the region $0 \le r < L$, $0 < \theta < \pi/2$, subject to the boundary conditions

$$\begin{aligned} u(r,0) &= 0, \qquad u(r,\pi/2) = 0, \quad 0 \le r < L \\ u(L,\theta) &= 1, \qquad 0 < \theta < \pi/2. \end{aligned}$$

(b) Use the change of variables $(x,y) \to (\xi,\eta)$ where $\xi = x + y$ and $\eta = x + 2y$ to show that the general solution to the partial differential equation

$$2u_{xx} - 3u_{xy} + u_{yy} = y, \qquad -\infty < x < \infty, \quad y \ge 0$$

is of the form

$$u(x,y) = f(x,y) + g_1(x+y) + g_2(x+2y),$$

where g_1 and g_2 are arbitrary functions (you must state the function $f(x,y)$ explicitly). Find the particular solution to this equation when $u(x,y)$ is subject to the boundary conditions $u(x,0) = 0$ and $u_y(x,0) = 0$.