APPLIED MATH PRELIMINARY EXAM

Please answer four questions on part A and three questions on part B. All questions are weighted evenly. Please provide clear and complete explanations of all steps taken, and make sure to justify any assumptions you make in the process. Good luck!

A. Core material

Answer four of the following six questions.

Problem A.1. Let V be an inner product space over $\mathbb{R}$ and $T : V \to V$ an orthogonal linear transformation.

(a) Show that every eigenvalue of T has absolute value 1.
(b) Suppose that W is a T-invariant subspace of V. Show that the orthogonal complement of W is T-invariant.

Problem A.2. Find an explicit formula for the entries of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$ in term of n

Problem A.3. Let f be a real-valued *bounded* monotonic function on the interval $(0, 1)$. Show that if f is continuous on $(0, 1)$, then it is also uniformly continuous there.

Problem A.4. Let K be a compact metric space and let $\{f_n\}$ be a sequence of real-valued continuous functions on K that converges uniformly to a function f on K. Show that:

(a) f is continuous on K.
(b) $\{f_n\}$ is bounded and equicontinuous on K.

Problem A.5. Suppose f is an entire function such that $|f(z)| \leq |\exp(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant C such that $f(z) = C\exp(z)$ for all $z \in \mathbb{C}$.

Problem A.6. Let a be a positive real number. Use the Calculus of Residues to show that $\displaystyle\int_0^\infty \frac{x^2}{(x^2 + a^2)^2} dx = \frac{\pi}{4a}$.

Date: 9:30–12:30 on Dec 21, 2006 in HH/410.

Answer three of the following four questions.

Problem B.1. Consider the initial–boundary value problem for the heat equation:

$$\begin{cases} u_t = u_{xx}, \qquad 0 < x < 1, \ t > 0, \\ u(x,0) = f(x), \qquad 0 < x < 1, \\ u(0,t) = 0, \qquad u_x(1,t) = \alpha u(1,t), \qquad t > 0, \end{cases}$$

where α is a real constant.

If $u(x,t)$ is a (C^2) solution, determine for which values of α the quantity

$$I(t) = \int_0^1 [u(x,t)]^2 \, dx$$

is a decreasing function of t. Prove that for such α the solution is *unique* for each given initial function f.

Problem B.2. Consider a function $u(r,t)$ of two variables r and t which solves the spherical wave equation:

$$u_{tt} = u_{rr} + \frac{2}{r} u_r.$$

(a) Find the general solution to the spherical wave equation by first using the change of variable $v = ru$.

(b) Solve the initial-value problem with $u(r,0) = 0$ and $u_t(r,0) = h(r)$, where h is a smooth, <u>even</u> function of r, $h(-r) = h(r)$.

(c) Is the solution $u(r,t)$ of the initial-value problem continuous at $r = 0$?

Problem B.3. (a) State the *Poincaré-Bendixson Theorem.*

(b) Prove that the planar system

$$\begin{aligned} x' &= x + y - x(x^2 + y^2) + \frac{1}{2}\sqrt{x^2 + y^2}, \\ y' &= -x + y - y(x^2 + y^2) \end{aligned}$$

has at least one nontrivial periodic orbit in the phase plane.
(Hint: Transform to polar coordinates.)

Problem B.4. Consider the family of nonlinear oscillator equations,

$$x'' + \epsilon x' + (x')^3 + x - x^5 = 0$$

for $\epsilon \geq 0$ a parameter.

(a) First assume $\epsilon > 0$. Find all equilibrium points and determine their stability. Then draw a phase portrait in the (x, x')-plane. Indicate stable and unstable directions for all saddle points and the correct direction for centers and spirals.

(b) Now consider $\epsilon = 0$. How are your responses from part (a) to be modified? Justify your answer.