APPLIED MATH PRELIMINARY EXAM

Please answer four questions on part A and three questions on part B. All questions are weighted evenly. Please provide clear and complete explanations of all steps taken, and make sure to justify any assumptions you make in the process. Good luck!

A. Core material

Answer four of the following six questions.

Problem A.1. Let

$$A = \begin{bmatrix} 8 & 6 & -20 \\ -3 & -1 & 16 \\ 0 & 0 & 4 \end{bmatrix}$$

Consider the inner product on $\mathbb{R}^3$ given by

$$\langle v, w\rangle = v^T A w,$$

where $v, w \in \mathbb{R}^3$ are viewed as column vectors and the superscript T denotes the transpose. Find a basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ and numbers $c_1, c_2, c_3 \in \mathbb{R}^3$ such that if $v = a_1v_1 + a_2v_2 + a_3v_3$ and $w = b_1v_1 + b_2v_2 + b_3v_3$ for some $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}$, then

$$\langle v, w\rangle = a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3.$$

Problem A.2. Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

(a) Determine the Jordan canonical form of M.
(b) For each positive integer n, determine the Jordan canonical form of M^n.

Problem A.3. Consider the mapping $\mathbf{f} = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{f}(x, y) = (x + y, 2\,x\,y), \quad (x, y) \in \mathbb{R}^2.$$

Show that the mapping $\mathbf{f}$ is locally invertible at the point $(x_0, y_0) = (2, -1)$. Find an explicit formula for its local inverse $\mathbf{g}(u, v)$ defined in a neighborhood of $\mathbf{f}(2, -1) = (1, -4)$ and compute $\mathbf{g}'(1, -4)$.

Problem A.4. Given a metric space (X, d) and a subset $A \subset X$ with $A \neq \emptyset$, define

$$\rho(x) = \inf_{z\in A} d(x, z).$$

Show that

$$|\rho(x) - \rho(y)| \le d(x, y), \quad \text{for all } x, y \in X.$$

Problem A.5. Use the Calculus of Residues to evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}dx$.

Problem A.6. Let $B = \{z \in \mathbb{C} \mid |z| \le 1\}$ be the closed unit disk. Suppose $f : B \to \mathbb{C}$ is a continuous function which is analytic in the interior of B. Furthermore, suppose $f(0) = \frac{1}{4}$ and $|f(\exp(it))| = \frac{5}{4} - \cos(t)$ for $0 \le t \le 2\pi$. Show that f has a zero in the interior of B.

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B. Applied math

Answer three of the following four questions.

Problem B.1. Consider the system of three differential equations,

$$\begin{aligned} \dot{x} &= -y + xz, \\ \dot{y} &= x + yz, \\ \dot{z} &= -z - x^2 - y^2 + z^2. \end{aligned}$$

(i) Find all critical points of the system.
(ii) For each critical point, construct the matrix of linearization, compute its eigenvalues, and classify the type of the critical point.
(iii) Approximate the center manifold near the origin $x = y = z = 0$ with Taylor series at the leading order and conclude whether the origin is stable, asymptotically stable or unstable.

Problem B.2. Consider the system of two differential equations

$$\begin{aligned} \dot{x} &= x - y - x^3, \\ \dot{y} &= x + y - y^3. \end{aligned}$$

(i) Rewrite the system of equations in polar coordinates (r, θ) and conclude stability of the origin $x = y = 0$.
(ii) Use the Poincare-Bendixson theorem to show that at least one limit cycle solution exists in the domain $1 \leq |\mathbf{x}| \leq \sqrt{2}$.
(iii) Use the Dulac–Bendixson criteria to show that at most one limit cycle solution exists in the domain $1 < |\mathbf{x}| < \sqrt{2}$.

Problem B.3. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad x \in [0, 1],$$

subject to the boundary conditions

$$\phi(0) - \phi'(0) = 0, \qquad \phi(1) + \phi'(1) = 0.$$

(i) Use Green's identities and show that $\lambda \geq 0$. Why is $\lambda > 0$?
(ii) Prove orthogonality of eigenfunctions corresponding to different eigenvalues.
(iii) Convert the problem of finding λ to the root finding problem for the transcendental equation

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}$$

Determine the eigenvalues graphically.

Problem B.4. Consider the heat equation in radial coordinate

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, \qquad r \in [0, a],$$

subject to the boundary condition

$$u\Big|_{r=a} = 0, \qquad t \geq 0,$$

and the initial condition

$$u\Big|_{t=0} = f(r), \qquad r \in [0, a].$$

(i) Find a general solution of the PDE problem by using separation of variables. Use the Bessel's equation

$$J_0''(z) + \frac{1}{z}J_0'(z) + J_0(z) = 0, \qquad z > 0$$

and the fact that the Bessel function $J_0(z)$ has infinitely many zeros $\{z_n\}_{n=1}^{\infty}$ on $z > 0$.

(ii) Represent parameters of the general solution in terms of integrals of the Bessel function $J_0(z)$ for $f(r) = 1$.

(iii) Analyze the behavior of $u(r, \theta, t)$ as $t \to \infty$.