On the main conjecture for a self-dual Hida family

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Abstract

We prove a statement towards the main conjecture of Iwasawa theory for the self-dual twist of the Galois representation attached to a Hida family. A key part of our argument is an application, for the first time, of the Euler system theory for Galois representations on modules over rings *not* assumed to be regular. Assuming some conjectures, we show that certain Selmer groups are torsion modules, and describe their associated divisors in terms of a suitable p-adic L-function, up to an error supported at an explicit finite set of height 1 primes.

1 Introduction

1.1. This paper studies the Iwasawa theory of certain Galois representations $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}(T)$, where T is a rank-2 free module over a ring $\mathbf{H}$ finite and flat over $\mathbf{Z}_p[\![T]\!]$ arising from the ordinary p-adic Hecke algebras constructed by Hida. The representations ρ are self-dual twists of the representations attached to p-adic families of ordinary newforms as considered in Hida [6] and Wiles [21]. One motivation for studying these *self-dual* representations is that the corresponding p-adic L-function interpolates *central* values of the associated classical L-functions for the modular forms appearing in the Hida family, cf. 2.16.

1.2. We study "main conjectures" for the representations ρ. Such a conjecture equates the divisorial support of a Selmer group attached to ρ (as defined in 2.9) with the divisor associated to a p-adic L-function interpolating special values of classical L-functions of modular forms. In the "sign -1" case, the Selmer group is not a torsion module, and we show that it has rank 1. (We hope in a future paper to identify its torsion submodule; cf. 4.13.) The p-adic L-function we consider here is that constructed by Kitagawa [10], and can be related to the Euler system of Kato by work of Ochiai [12]. What we show below is how these main conjectures follow from various non-vanishing conjectures. See Theorem 2.21 for a precise statement of the main result.

1.3. Many of the results presented here are similar to those in [14], the main difference being that we do not assume our Hecke algebras are power series rings. In fact, a major goal of this paper is to serve as a kind of case study in extending the theory of Euler systems to normal rings. Our method can be viewed as an extension of the method developed in [13] for power series rings. We have chosen to focus on the 1-variable context provided by self-dual twists in part out of a desire to simplify this Euler system argument and in part to minimize the overlap with [14]. As we will discuss in a future paper, the methods we present here apply to quite general situations, including a direct generalization of the 2-variable results which are the focus of [14].

1.4. The basic idea of our method is as follows. Assuming the Euler system is non-trivial, almost all specializations of the Hida family should have finite strict Selmer group (equivalently, finite Shafarevich-Tate group) whose order can be bounded by the standard Euler system machinery in terms of (generically non-zero) special values of the p-adic L-function. Fixing a prime $\mathfrak{p}$ in the support of the Selmer group of the family, we show how the asymptotic behavior of the size of the Selmer groups associated to nearby specializations determines the $\mathfrak{p}$-length of the Selmer group of the family. The key results which make this argument work are control theorems with a *uniform* bound on the error, cf. §3. In order to bound the size of the Selmer group from below in terms of the p-adic L-function, we appeal a bound from the 2-variable main conjecture, of which a proof in some cases has been announced by Skinner-Urban (cf. [20]), and descend to the self-dual line.

1.5. The main drawback of our approach is that we must exclude a finite set of primes. These primes are:

(1.5.1) the prime of $\mathbf{H}$ over (p), i.e., we do not study the "μ-invariant",

(1.5.2) primes of $\mathbf{H}$ not dividing the largest ideal $\mathfrak{c} \subseteq \Lambda$ such that $\operatorname{Im} \rho$ contains $\mathrm{SL}_2(\Lambda) \cap (1 + \mathfrak{c} \cdot \mathrm{M}_2(\Lambda))$,

(1.5.3) primes of $\mathbf{H}$ which are *ramified* over $\Lambda \subseteq \mathbf{H}$, and

(1.5.4) primes in the support of $\mathrm{H}^2(\Omega/\mathbf{Q}, T \oplus T^*)_{\mathbf{H}\text{-tors}}$, $\Omega = \mathbf{Q}(\mu_{p^\infty}, T)$.

As our method involves passing to the associated rigid analytic space (albeit in an ad-hoc manner, and not explicitly), no information about the μ-invariant is preserved. We impose below the assumption that $\mathbf{H}$ corresponds to a non-CM family, so the existence of the ideal $\mathfrak{c}$ in (1.5.2) is a theorem of Hida, see [8, Thm. I]. If $p \geq 7$ and $\operatorname{Im} \overline{\rho}$ contains $\mathrm{SL}_2(\mathbf{F}_p)$, then $\mathfrak{c} = 1$. The primes (1.5.4) must be excluded in order to avoid the error terms in the Euler system bound. Denoting by $\overline{\rho}$ the residual representation on $T/\mathfrak{m}T$, where $\mathfrak{m} \subseteq \mathbf{H}$ is the maximal ideal, this restriction can be ignored, e.g., as we show, in the case that $\overline{\rho}$ contains $\mathrm{SL}_2(\mathbf{F}_p)$. The ramified primes (1.5.3), on the other hand, only cause a problem in one step of the argument: 3.14. It might be possible to circumvent this problem through sufficiently clever analysis of the intervening Galois cohomology.

2 Notations

2.1. In this section, we give definitions and notation, as well as results of Ochiai and others and related conjectures which we need to apply our method. The statement of our main theorem is given in 2.21.

We denote by $\overline{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ and choose isomorphisms $\mathbf{C} \cong \mathbf{C}_p$ for every prime p (note that this determines embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$). In §3, we define the term *rational prime*, by which we do *not* mean a prime element of $\mathbf{Z}$. Non-zero prime elements of $\mathbf{Z}$ are referred to as *prime numbers*.

In order to make our notation consistent with the papers of Ochiai, [12, 13, 14], for a prime number ℓ, we denote by $\mathrm{Frob}_\ell \in G_{\mathbf{Q}}$ a *geometric* Frobenius element and normalize our Galois representations accordingly (i.e., our representations attached to modular forms are dual to those considered by authors for whom Frob_ℓ denotes an arithmetic Frobenius element). We denote by $\omega : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$, resp. $\chi_{\mathrm{cyc}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$, the Teichmüller, resp. p-adic cyclotomic, character giving the action of $G_{\mathbf{Q}}$ on the pth, resp. p-power, roots of unity.

If R is a (commutative) ring, M is a finitely-generated R-module, R' is a flat extension of R, and $\mathfrak{p} \subseteq R'$ is a prime ideal, we define the $\mathfrak{p}$*-length* $\mathrm{lgth}_{\mathfrak{p}} M$ of M as the length of $(M \otimes_R R')_{\mathfrak{p}}$ as an $R'_{\mathfrak{p}}$-module, i.e.,

$$\mathrm{lgth}_{\mathfrak{p}} M = \mathrm{lgth}_{R'_{\mathfrak{p}}} (M \otimes_R R')_{\mathfrak{p}}.$$

Note that if $\mathfrak{p}R''$ is prime for R'' a flat extension of R', then $\mathrm{lgth}_{\mathfrak{p}} M = \mathrm{lgth}_{\mathfrak{p}R''} M$, so we may abuse notation and use $\mathrm{lgth}_{\mathfrak{p}} M$ to denote both lengths.

2.2. Choose an integer N, a prime $p > 2$ not dividing N, and a finite extension F of $\mathbf{Q}_p$. Let $f \in S_k(N, \psi_f; \mathcal{O}_F)$ be a normalized cuspidal newform on $\Gamma_1(N)$ of even weight $k \geq 2$, character ψ_f, and with Fourier coefficients in the integer ring $\mathcal{O}_F$ of F. We assume moreover that f is *p-ordinary* in the sense that the pth Fourier coefficient $a_p(f)$ of f is a unit in $\mathcal{O}_F$. By work of Deligne [2], to any such f is associated a continuous Galois representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O}_F)$ unramified outside Np which satisfies

$$\mathrm{trace}\, \rho_f(\mathrm{Frob}_\ell) = a_\ell(f), \quad \det \rho_f(\mathrm{Frob}_\ell) = \psi_f(\ell)\ell^{k-1}$$

for any prime number $\ell \nmid Np$.

2.3. Hida [7] has shown that every modular form f as above can be put into an ordinary p-adic family $\mathcal{F}$ of modular forms of Nebentypus $\psi_{\mathcal{F}} = \psi_f \omega^{k-2}$ in a sense which we now describe. The form $\mathcal{F}$ can be considered as a formal q-expansion $\mathcal{F} = \sum_{n=1}^{\infty} a_n(\mathcal{F}) q^n$ whose coefficients belong to a *normal* local domain $\mathbf{H} = \mathbf{H}_{\mathcal{F}}$ finite and flat over $\Lambda := \mathbf{Z}_p[\![\Gamma_D]\!]$, where $\Gamma_D \cong \mathbf{Z}_p$ is the p-part of the group of diamond operators acting on the tower of modular curves $\{Y_1(p^n)\}_n$. The ring $\mathbf{H}$ we work with here is the *integral closure* of the quotient of Hida's p-adic Hecke algebra by the minimal prime corresponding to $\mathcal{F}$. Denote by

$$\chi_D : \Gamma_D \xrightarrow{\cong} 1 + p\mathbf{Z}_p$$

the canonical character. We say that a character $\kappa \in \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{H}, \mathbf{C}_p)$ is *arithmetic* if the character

$$\psi_\kappa = \kappa|_{\Gamma_D} \cdot \chi_D^{-w(\kappa)} : \Gamma_D \longrightarrow \mathbf{C}_p^\times$$

has finite order for some (uniquely determined) integer $w(\kappa)$, called the *weight* of κ. Let $\Gamma_C = \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$, where $\mathbf{Q}_\infty$ is the $\mathbf{Z}_p$-extension of $\mathbf{Q}$, and denote by

$$\chi_C : \Gamma_C \xrightarrow{\cong} 1 + p\mathbf{Z}_p$$

the character $\omega^{-1}\chi_{\mathrm{cyc}}$. The canonical isomorphism $\Gamma_C \cong \Gamma_D$ determined by χ_C and (the inverse of) χ_D allows us to consider, by abuse of notation, ψ_κ as a character of Γ_C, i.e., a Dirichlet character of p-power order and p-power conductor. Define $p^{r(\kappa)}$ to be the conductor of ψ_κ. If κ is of weight $w(\kappa) \geq 0$, then the specialization f_κ of $\mathcal{F}$ at κ is a p-stabilized eigenform of level $Np^{r(\kappa)}$, weight $w(\kappa) + 2$, and Nebentypus character $\psi_{\mathcal{F}}\psi_\kappa\omega^{-w(\kappa)}$. That the form f belongs to $\mathcal{F}$ means that the p-stabilized eigenform associated to f is the specialization of $\mathcal{F}$ at an arithmetic character κ of weight $k - 2$.

2.4. Let $\mathbf{K} = \mathrm{Frac}\,\mathbf{H}$. To $\mathcal{F}$ is associated a continuous (in the sense of Hida) Galois representation $\rho_{\mathcal{F}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{K})$ which is unramified outside of Np and has

$$\mathrm{trace}\,\rho_{\mathcal{F}}(\mathrm{Frob}_\ell) = a_\ell(\mathcal{F}), \quad \det\rho_{\mathcal{F}}(\mathrm{Frob}_\ell) = \psi_{\mathcal{F}}(\ell)\langle\ell\rangle\ell,$$

where $\langle\ell\rangle$ is the image of $\ell \in \mathbf{Z}_p^\times$ under the composition $\mathbf{Z}_p^\times \twoheadrightarrow 1 + p\mathbf{Z}_p \xleftarrow{\cong} \Gamma_D \hookrightarrow \mathbf{K}^\times$, the second arrow being given by χ_D. In particular, there is a $G_{\mathbf{Q}}$-stable lattice $T_{\mathcal{F}} \subseteq \mathbf{K}^2$, i.e., $T_{\mathcal{F}}$ is finitely-generated and of generic rank 2 over $\mathbf{H}$. In what follows, we assume the family $\mathcal{F}$ does not have CM and that the residual representation of ρ_f is *absolutely irreducible*. We also assume that the $G_{\mathbf{Q}}$-stable lattice $T_{\mathcal{F}}$ is *free* of rank 2 over $\mathbf{H}$; such a choice of free lattice can always be made under the additional assumption that the residual representation of ρ_f is *p-distinguished*, cf. [11, Cor. 2.6]. (Note that *ibid* claims, incorrectly, that the existence of a free lattice is guaranteed by irreducibility of $\overline{\rho}_f$. We thank Olivier Fouquet for pointing this out to us.) We may thus consider $\rho_{\mathcal{F}}$ as a representation into $\mathrm{GL}_2(\mathbf{H})$. The specialization of $\rho_{\mathcal{F}}$ at an arithmetic character κ, by which we mean the $G_{\mathbf{Q}}$-module $T_{\mathcal{F}} \otimes_{\mathbf{H}} \kappa(\mathbf{H})$, is isomorphic to ρ_{f_κ} after extending scalars from $\kappa(\mathbf{H})$ to $\overline{\mathbf{Q}}_p$.

2.5. We extend the above setup by deforming in the cyclotomic direction. A character $\lambda : \Gamma_C \to \overline{\mathbf{Q}}_p$ is said to be *arithmetic* if $\lambda \cdot \chi_C^{-w(\lambda)}$ has finite order for some (uniquely determined) integer $w(\lambda)$, the *weight* of λ. Define the *nearly-ordinary Hecke algebra* associated to $\mathcal{F}$ to be the ring

$$\Lambda_{\mathbf{H}} = \mathbf{H} \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{Z}_p[\![\Gamma_C]\!] = \mathbf{H}[\![\Gamma_C]\!],$$

which is a local domain finite and flat over $\mathbf{Z}_p[\![\Gamma_C \times \Gamma_D]\!]$. This ring plays the role of a 2-variable Iwasawa algebra. We define an *arithmetic specialization* of $\Lambda_{\mathbf{H}}$ to be a pair (λ, κ) where λ is an arithmetic character of Γ_C and κ is an arithmetic character of $\mathbf{H}$. Any arithmetic specialization determines a homomorphism $(\lambda, \kappa) : \Lambda_{\mathbf{H}} \to \overline{\mathbf{Q}}_p$: (λ, κ) is the unique continuous homomorphism which is equal to κ when restricted to $\mathbf{H}$ and which takes the value $\lambda(\gamma)$ when evaluated at an element $\gamma \in \Gamma_C$. We denote by $\mathfrak{p}_{(\lambda,\kappa)}$ the kernel of (λ, κ). For ease of notation, we set $(j, \kappa) = (\chi_C^j, \kappa)$. Given a finitely-generated $\Lambda_{\mathbf{H}}$-module M and any pair (λ, κ) of continuous characters (of Γ_C and $\mathbf{H}$, respectively), we define the *specialization* of M at (λ, κ) to be the $\Lambda_{\mathbf{H}}/\mathfrak{p}_{(\lambda,\kappa)}$-module $M_{(\lambda,\kappa)} = M/\mathfrak{p}_{(\lambda,\kappa)}M$.

There is again a Galois representation $\rho_{\mathcal{F}}^{\mathrm{no}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\Lambda_{\mathbf{H}})$ obtained by the formula

$$\rho_{\mathcal{F}}^{\mathrm{no}} = \rho_{\mathcal{F}} \otimes \chi^{\mathrm{univ}},$$

where χ^{univ} is the universal character $\chi^{\mathrm{univ}} : G_{\mathbf{Q}} \twoheadrightarrow \Gamma_C \to \mathbf{Z}_p[\![\Gamma_C]\!]^\times$, the second arrow being given by $\gamma \mapsto \gamma$. Let T^{no} be the representation space of $\rho_{\mathcal{F}}^{\mathrm{no}}$. The specialization of $\rho_{\mathcal{F}}^{\mathrm{no}}$ at an arithmetic point (λ, κ) with $w(\kappa) \geq 0$ is isomorphic to $\rho_{f_\kappa} \otimes \lambda$ after extending scalars to $\overline{\mathbf{Q}}_p$. Wiles [21] has shown that there is a filtration $T^{\mathrm{no}} \supseteq \mathrm{F}^+ T^{\mathrm{no}} \supseteq 0$ of T^{no} by $\Lambda_{\mathbf{H}}[G_{\mathbf{Q}_p}]$-submodules such that $\mathrm{F}^+ T^{\mathrm{no}}$ and $\mathrm{F}^- T^{\mathrm{no}} = T^{\mathrm{no}}/\mathrm{F}^+ T^{\mathrm{no}}$ are free $\Lambda_{\mathbf{H}}$-modules of rank 1. The action of $G_{\mathbf{Q}_p}$ on $\mathrm{F}^+ T^{\mathrm{no}}$ is via the character $\alpha \otimes \chi^{\mathrm{univ}}$, where $\alpha : G_{\mathbf{Q}_p} \to \mathbf{H}^\times$ is an unramified character such that $\kappa\big(\alpha(\mathrm{Frob}_p)\big) = a_p(f_\kappa)$ for every arithmetic character κ of weight $w(\kappa) \geq 0$.

2.6. We now discuss the quotient of $\Lambda_{\mathbf{H}}$ which is the primary focus of our investigations. As we are assuming $p > 2$, the element $\gamma_D \in \Gamma_D$ mapping to $p + 1$ under χ_D admits a unique square root $\gamma_D^{1/2}$. We define an element

$$\Theta = \big(\gamma_C - \gamma_D^{1/2} \cdot (p+1)\big) \subseteq \mathbf{Z}_p[\![\Gamma_C \times \Gamma_D]\!],$$

where γ_C is the element of Γ_C mapping to $p + 1$ under χ_C. (In the CM case, Θ is the "critical divisor" of Greenberg [3].) Abusing notation, we often write Θ again for the ideal $\Theta\Lambda_{\mathbf{H}} \subseteq \Lambda_{\mathbf{H}}$. In what follows, we identify $\Lambda_{\mathbf{H}}/\Theta$ with $\mathbf{H}$ via the composition $\mathbf{H} \hookrightarrow \Lambda_{\mathbf{H}} \twoheadrightarrow \Lambda_{\mathbf{H}}/\Theta$. For any $\mathbf{Z}_p[\![\Gamma_C \times \Gamma_D]\!]$-module M, set $M_\Theta = M/\Theta M$. One can alternately describe the $G_{\mathbf{Q}}$-module $T := T_\Theta^{\mathrm{no}}$ as the representation space of $\rho := \rho_{\mathcal{F}} \otimes \theta$, where $\theta : G_{\mathbf{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \to \mathbf{H}^\times$ is the continuous character satisfying $\theta(\gamma_C) = \gamma_D^{1/2}(p+1)$; thus, as we expose below (in 2.16), T can be considered as the representation space of both the "self-dual twist" of $\rho_{\mathcal{F}}$ and the "self-dual quotient" of $\rho_{\mathcal{F}}^{\mathrm{no}}$.

2.7. A torsion $\Lambda_{\mathbf{H}}$- or $\mathbf{H}$-module M is said to be *pseudo-null* if it is supported in codimension 2 or greater. Note that an $\mathbf{H}$-module is pseudo-null if and only if it contains only finitely many elements. A homomorphism is said to be a *pseudo-isomorphism* if its kernel and cokernel are pseudo-null. For any $\mathbf{Z}_p$-module M, we denote by $M^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(M, \mathbf{Q}_p/\mathbf{Z}_p)$ its Pontryagin dual. For $R = \Lambda_{\mathbf{H}}$, $\mathbf{H}$, or $\mathbf{Z}_p$, if M is a finitely-generated R-module with $G_{\mathbf{Q}}$-action, we denote by M^* its *Tate dual* $G_{\mathbf{Q}}$-module $M^* = \mathrm{Hom}_R\big(M, R(1)\big)$.

Set $D^{\mathrm{no}} = (\Lambda_{\mathbf{H}})^\vee$ and $D = \mathbf{H}^\vee$, viewed as discrete trivial $G_{\mathbf{Q}}$-modules. We define $W^{\mathrm{no}} = T^{\mathrm{no}} \otimes_{\Lambda_{\mathbf{H}}} D^{\mathrm{no}}$, resp. $W^{\mathrm{no},*} = T^{\mathrm{no},*} \otimes_{\Lambda_{\mathbf{H}}} D^{\mathrm{no}}$, to be the discrete modules associated to T^{no}, resp. $T^{\mathrm{no},*}$ and make the analogous definition for W and W^*. The filtration on T^{no} induces in an obvious way a filtration on W^{no}, the positive part of which we denote by $\mathrm{F}^+ W^{\mathrm{no}}$. There is also an induced filtration on $T^{\mathrm{no},*}$ given by $\mathrm{F}^+(T^{\mathrm{no},*}) = (\mathrm{F}^- T^{\mathrm{no}})^* = \mathrm{Hom}_{\Lambda_{\mathbf{H}}}\big(\mathrm{F}^- T^{\mathrm{no}}, \Lambda_{\mathbf{H}}(1)\big)$, which in turn induces a filtration on $W^{\mathrm{no},*}$. Note that we have $(T^{\mathrm{no},*})_\Theta = T^* := \mathrm{Hom}_{\mathbf{H}}\big(T, \mathbf{H}(1)\big)$ and $(T^{\mathrm{no},*})_{(\lambda,\kappa)} = (T^{\mathrm{no}}_{(\lambda,\kappa)})^*$ for any arithmetic specialization (λ, κ). We set $V_{(\lambda,\kappa)} = T^{\mathrm{no}}_{(\lambda,\kappa)} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ and $V^*_{(\lambda,\kappa)} = (T^{\mathrm{no},*})_{(\lambda,\kappa)} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. All these modules have filtrations induced in the obvious way from the filtrations on T^{no} and $(T^{\mathrm{no}})^*$. Note that $W^{\mathrm{no}}[\Theta] = W = (T_{\mathcal{F}} \otimes \theta) \otimes_{\mathbf{H}} D$.

2.8. One can axiomatize the above setup. Let T be a finite free Λ-module with continuous $G_{\mathbf{Q}}$ action satisfying the following hypotheses:

(2.8.1) T is unramified outside of a finite set $\Sigma \ni p$ of primes,

(2.8.2) $\mathrm{H}^0(\mathbf{Q}, T/\mathfrak{m}_\Lambda T) = 0$,

(2.8.3) there is a $G_{\mathbf{Q}_p}$-stable filtration $T \supseteq \mathrm{F}^+ T \supseteq 0$ with $\mathrm{F}^+ T$ and $\mathrm{F}^- T = T/\mathrm{F}^+ T$ free over Λ, and

(2.8.4) for any finite extension $K/\mathbf{Q}$ and any prime v of K, $T(K_v) = T^*(K_v) = 0$; if $v \mid p$, then $(\mathrm{F}^+ T^*)(K_v) = 0$.

Set moreover $W = T \otimes_\Lambda \Lambda^\vee$. This notation does not conflict with our previously established notation, in that we mean to apply the discussion of those two paragraphs to the module W as defined above. To do so, we may consider any $\mathbf{H}$-module also as a Λ-module by our identification with Λ with the subring $\mathbf{Z}_p[\![\Gamma_D]\!] \subseteq \mathbf{H}$.

2.9. We now introduce local conditions and Selmer groups for the Galois representations defined above. Let K be a number field and let $\Sigma = \Sigma_K$ be the finite set of places of K containing all places which either lie over $p\infty$ or are ramified for T. If M is a finitely-generated module over R for $R = \Lambda_{\mathbf{H}}$, $\mathbf{H}$, or $\mathbf{Z}_p$, then we define

$$\mathrm{H}^1(K, M) = \varprojlim_n \mathrm{H}^1(K, M/\mathfrak{m}_R^n M)$$
$$\mathrm{H}^1(K_v, M) = \varprojlim_n \mathrm{H}^1(K_v, M/\mathfrak{m}_R^n M)$$

for every place v of K, where $\mathfrak{m}_R$ is the maximal ideal of R. If M is cofinitely-generated over one of these rings (and discretely topologized), the groups $\mathrm{H}^1(K, M)$ and $\mathrm{H}^1(K_v, M)$ denote the usual (continuous) cohomology groups.

2.10. For any ring R, R-module M, and submodule $N \subseteq M$, define the *R-saturation* of N in M as the submodule $N^{R\text{-sat}} = \phi^{-1}\phi(N)$, where $\phi : M \to M \otimes_R \operatorname{Frac} R$ is the natural map and $\operatorname{Frac} R$ denotes the total ring of fractions of R. We only use this definition when $R = \Lambda_{\mathbf{H}}$, $\mathbf{H}$, or $\mathbf{Z}_p$ and M is a finitely-generated R-module; this allows us to drop the R from the notation, as the ring can always be determined from the context. Note that N^{sat} always contains $M_{R\text{-tors}}$. We review here the Selmer groups defined in [1] following Plater [17], Greenberg [4], and others for the $\Lambda_{\mathbf{H}}$-, $\mathbf{H}$-, and $\mathbf{Z}_p$-modules introduced above.

2.11. Suppose M is a finitely-generated R-module with G_K-action admitting, for each $v \mid p$, a G_{K_v}-stable filtration $M \supseteq \mathrm{F}_v^+ M \supseteq 0$. For $v \nmid p$, set

$$\mathrm{H}^1_{\mathrm{f}}(K_v, M) = \mathrm{H}^1_{\mathrm{ur}}(K_v, M)^{\mathrm{sat}} \subseteq \mathrm{H}^1(K_v, M)$$

and in the case $v \mid p$ set

$$\mathrm{H}^1_{\mathrm{f}}(K_v, M) = \ker\bigl(\mathrm{H}^1(K_v, M) \longrightarrow \mathrm{H}^1(I_v, F^- M)\bigr)^{\mathrm{sat}} \subseteq \mathrm{H}^1(K_v, M),$$

where I_v is the inertia subgroup of G_{K_v}. Recall that for every place v of K and every compact R-module M for $R = \Lambda_{\mathbf{H}}$, $\mathbf{H}$, or $\mathbf{Z}_p$, local Tate duality provides a perfect pairing

$$\mathrm{H}^1(K_v, M) \times \mathrm{H}^1\bigl(K_v, M^\vee(1)\bigr) \longrightarrow R^\vee.$$

Define, for any v, $\mathrm{H}^1_{\mathrm{f}}\bigl(K_v, M^\vee(1)\bigr)$ as the orthogonal complement of $\mathrm{H}^1_{\mathrm{f}}(K_v, M)$ under the Tate pairing.

2.12. For any G_K-module M for which the notation has been defined (compact or discrete), we set

$$\begin{aligned}
\mathrm{Sel}^\Sigma(K, M) &= \ker\Bigl(\mathrm{H}^1(K, M) \to \bigoplus_{v \notin \Sigma} \mathrm{H}^1_{\mathrm{s}}(K_v, M)\Bigr) \\
\mathrm{Sel}(K, M) &= \ker\Bigl(\mathrm{Sel}^S(K, M) \to \bigoplus_{v \in \Sigma} \mathrm{H}^1_{\mathrm{s}}(K_v, M)\Bigr) \\
\mathrm{Sel}_\Sigma(K, M) &= \ker\Bigl(\mathrm{Sel}(K, M) \to \bigoplus_{v \in \Sigma} \mathrm{H}^1(K_v, M)\Bigr),
\end{aligned}$$

where $\mathrm{H}^1_{\mathrm{s}}(K_v, M) = \mathrm{H}^1(K_v, M)/\mathrm{H}^1_{\mathrm{f}}(K_v, M)$. If $K = \mathbf{Q}$, then we suppress K from the notation, i.e., $\mathrm{Sel}(M) = \mathrm{Sel}(\mathbf{Q}, M)$, etc. It is worthwhile to remark here that we could give a simpler, though *ad hoc*, definition of these Selmer groups in the case $M = T$, T^*, W, etc. using the fact (cf. [1, Lemma 2.9]) that $\mathrm{H}^1(\mathbf{Q}_v, T)$ and $\mathrm{H}^1(\mathbf{Q}_v, T^*)$ are $\mathbf{H}$-torsion for $v \nmid p$.

2.13 Kato's Euler system. If $f = \sum_{n=1}^{\infty} a_n(f)q^n$ is a normalized cuspidal newform of weight k and character ψ_f, there is attached an L-function

$$L(f,s) = \sum_{n=1}^{\infty} a_n(f) n^{-s}$$

which has analytic continuation to all of $\mathbf{C}$ and satisfies a functional equation relating $L(f,s)$ and $L(\overline{f}, k-s)$, where $\overline{f}$ is the form (of weight k and character $\overline{\psi}_f$) obtained by applying complex conjugation to the Fourier coefficients of f. Thus, if $f = \overline{f}$, then it makes sense to speak of the *sign* $\varepsilon_f = \pm 1$ in the functional equation of f; the sign is equal to the parity of the order of vanishing of $L(f,s)$ at $s = k/2$. The values of $L(f,s)$ at integer arguments are related via an interpolation property to an Euler system constructed by Kato. This Euler system arises from elements in the K_2 groups of modular curves via the Chern class map and comprises elements of the Galois cohomology of the dual representation T^* over cyclotomic extensions of $\mathbf{Q}$.

Suppose an eigenform f as above has coefficients in $\mathbf{Q}_f$, a finite extension of $\mathbf{Q}$. Attached to f is a de Rham realization $\mathrm{V}_{\mathrm{dR}}(f)$, which is a 2-dimensional $\mathbf{Q}_f$-vector space equipped with a decreasing filtration $\mathrm{F}^i \mathrm{V}_{\mathrm{dR}}(f)$. This filtration satisfies: $\mathrm{F}^0 \mathrm{V}_{\mathrm{dR}}(f) = \mathrm{V}_{\mathrm{dR}}(f)$, $\mathrm{F}^{k+2} \mathrm{V}_{\mathrm{dR}}(f) = 0$, and $\mathrm{F}^i \mathrm{V}_{\mathrm{dR}}(f) \cong \mathbf{Q}_f \cdot f$ canonically for $0 < i < k+2$.

2.14 Theorem (Kato [9]). *Let R be the set of squarefree natural numbers prime to Np. With notation as in 2.2–2.7, there are elements $z(r) \in \mathrm{H}^1(\mathbf{Q}(\zeta_r), T^{\mathrm{no},*})$ for every $r \in R$ such that if $q \in R$ does not divide r, then*

$$\mathrm{Nm}_{\mathbf{Q}(\zeta_{rq})/\mathbf{Q}(\zeta_r)} z(rq) = P_q(\mathrm{Frob}_q) z(r),$$

where $P_q(X) = \det(1 - \mathrm{Frob}_q X \mid T^{\mathrm{no}}) \in \Lambda_{\mathbf{H}}[X]$. Each $z(r)$ is moreover unramified at all primes not dividing Np.

For each arithmetic specialization (j,κ) such that $1 \leq j \leq w(\kappa)+1$, let $z(1)_{(j,\kappa)} \in \mathrm{H}^1(\mathbf{Q}, T^{\mathrm{no},*}_{(j,\kappa)})$ be the specialization of $z(1)$ via the character (j,κ). Then the image of $z(1)_{(j,\kappa)}$ under the localization map

$$\mathrm{loc}_{\mathrm{s}} : \mathrm{H}^1(\mathbf{Q}, T^{\mathrm{no},*}_{(j,\kappa)}) \longrightarrow \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^{\mathrm{no},*}_{(j,\kappa)})$$

satisfies

$$\exp^*\bigl(\mathrm{loc}_{\mathrm{s}}\, z(1)_{(j,\kappa)}\bigr) = c(j,\kappa) L_{(p)}\bigl(f_\kappa, j\bigr) \cdot \overline{\delta}_\kappa$$

as elements of

$$\mathrm{F}^{w(\kappa)+2-j}\, \mathrm{V}_{\mathrm{dR}}(\overline{f}_\kappa) \subseteq \mathrm{F}^0\, \mathrm{D}_{\mathrm{dR}}(V^*_{(j,\kappa)}),$$

where $\exp^* : \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^{\mathrm{no},*}_{(j,\kappa)}) \longrightarrow \mathrm{F}^0\, \mathrm{D}_{\mathrm{dR}}(V^*_{(j,\kappa)})$ *is the dual exponential map, $\overline{\delta}_\kappa$ is the basis of* $\mathrm{F}^{w(\kappa)+2-j}\, \mathrm{V}_{\mathrm{dR}}(\overline{f}_\kappa)$ *corresponding to $\overline{f}_\kappa$, and $c(j,\kappa)$ is an explicit non-zero constant (involving complex and p-adic periods).*

2.15. The precise values of the constants $c(j,\kappa)$ in Theorem 2.14 depend on certain conventions used to define the classes $z(r)$. We follow here the conventions of Ochiai [14, Thm. 6.11], which allow one to relate $z(r)$ to a p-adic L-function in a specific way, cf. Theorems 2.18 and 2.19 below.

By the theorem, the "base" class $z(1) \in \mathrm{H}^1(\mathbf{Q}, T^{\mathrm{no},*})$ belongs to the submodule $\mathrm{Sel}^{\Sigma}(T^{\mathrm{no},*})$: recall that this is the same as $\mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, T^{\mathrm{no},*})$. Set $Z^{\mathrm{no}} = \mathrm{Sel}^{\Sigma}(T^{\mathrm{no},*})/\langle z(1)\rangle$ and $Z = \mathrm{Sel}^{\Sigma}(T^*)/\langle z(1)_\Theta\rangle$, where $z(1)_\Theta$ denotes the image of $z(1)$ under the natural map $\mathrm{Sel}^{\Sigma}(T^{\mathrm{no},*}) \to \mathrm{Sel}^{\Sigma}(T^*)$ induced by the quotient $T^{\mathrm{no},*} \to T^{\mathrm{no},*}/\Theta T^{\mathrm{no},*} = T^*$.

2.16. Let g be a modular form of even weight $j \geq 2$ and character ω^{k-j} which arises as a specialization of the Hida family $\mathcal{F}$, say via the character $\kappa : \mathbf{H} \to \overline{\mathbf{Q}}_p$ of weight $w(\kappa) = j-2$. If $j \equiv k \pmod{p-1}$, then g has trivial character, the specialization $(k/2,\kappa) : \Lambda_{\mathbf{H}} \to \overline{\mathbf{Q}}_p$ factors through $\Lambda_{\mathbf{H}}/\Theta$, and $z(1)_{(k/2,\kappa)}$ interpolates (in the sense of Theorem 2.14) the central L-value $L(f,k/2)$. If $\varepsilon_f = \pm 1$ is the sign in the functional equation of f, then any modular form g of even weight j such that $j \equiv k \pmod{p-1}$ which arises as a specialization of $\mathcal{F}$ has $\varepsilon_g = \varepsilon_f$. We then say that ε_f is the *sign* of the Hida family $\mathcal{F}$, and we denote it by $\varepsilon_{\mathcal{F}}$. It is possible $\mathcal{F}$ might not have a sign; we often use the phrase "the sign of $\mathcal{F}$ is not -1", meaning that either $\mathcal{F}$ does not have a sign or if it does, then its sign is 1.

2.17. What follows are three statements that are fundamental to applying our method to the context of Hida families. The first is the existence of a 2-variable p-adic L-function attached to the Galois representation T^{no} of 2.5. The second is a construction of Ochiai [12] of a "Coleman" map from a local (at p) cohomology group of the Galois representation T^{no} to the ring $\Lambda_{\mathbf{H}}$; the image under this map of the cohomology class arising from Kato's Euler system attached to $\mathcal{F}$ is the 2-variable p-adic L-function (up to a unit in $\Lambda_{\mathbf{H}}$). We also describe an inequality which is "half" of the 2-variable main conjecture of Iwasawa theory for (the Pontryagin dual of) the Selmer group associated to the Galois module T^{no}.

2.18 Theorem (Kitagawa [10])**.** *There is an element $\mathcal{L} \in \Lambda_{\mathbf{H}}$ such that for every arithmetic weight (λ,κ) with $1 \leq w(\lambda) \leq w(\kappa)+1$, the interpolation formula*

$$(\lambda,\kappa)(\mathcal{L}) = d(\lambda,\kappa)L(f_\kappa, \omega^{-w(\lambda)}\lambda\chi_C^{-w(\lambda)}, w(\lambda))$$

holds, where $d(\lambda,\kappa)$ is an explicit non-zero constant (involving complex and p-adic periods).

2.19 Theorem (Ochiai [14, Cor. 6.17])**.** *There exists an injective $\Lambda_{\mathbf{H}}$-linear homomorphism*

$$\Xi : \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^{\mathrm{no},*}) \longrightarrow \Lambda_{\mathbf{H}}$$

with pseudo-null cokernel such that $\Xi\big(\mathrm{loc_s}\, z(1)\big)$ *and* $\mathcal{L}$ *generate the same ideal in* $\Lambda_{\mathbf{H}}$.

2.20 Conjecture. *For every prime ideal* $\mathfrak{p} \subseteq \Lambda_{\mathbf{H}}$ *of height* 1 *not containing* (p),

$$\mathrm{lgth}_{\mathfrak{p}}\, \mathrm{Sel}(W^{\mathrm{no}})^{\vee} \geq \mathrm{ord}_{\mathfrak{p}}\, \mathcal{L}.$$

Skinner-Urban have announced a proof of this inequality in some cases; see [20] for an overview.

2.21. The remainder of the paper will be devoted to proving the following result, which depends on the conjectural non-vanishing of the restriction of Kato's Euler system to the quotient $\Lambda_{\mathbf{H}}/\Theta \cong \mathbf{H}$ of $\Lambda_{\mathbf{H}}$, Conjectures 4.3 and 4.4. In the case that the representation ρ is not self-dual, we additionally assume non-degeneracy of the p-adic height pairing, Conjecture 4.5; this assumption should be unnecessary, but as our primary interest is in self-dual representations we do not pursue this point.

Theorem. *Assume Conjectures 4.3 and 4.4 and that either* ρ *is self-dual or Conjecture 4.5 holds. Let* $\mathfrak{p} \subseteq \mathbf{H}$ *be a prime of height* 1 *which is not of type* (1.5.1)–(1.5.4)*; if* $\overline{\rho}(G_{\mathbf{Q}}) \supseteq \mathrm{GL}_2(\mathbf{F}_p)$*, then we only need exclude primes of type* (1.5.1)–(1.5.3)*. If the sign of* $\mathcal{F}$ *is not* -1*, then* $\mathrm{Sel}(W)^{\vee}$ *is a torsion* $\mathbf{H}$*-module and we have the inequality*

$$\mathrm{lgth}_{\mathfrak{p}}\, \mathrm{Sel}(W)^{\vee} \leq \mathrm{ord}_{\mathfrak{p}}\, \mathcal{L}_{\Theta}. \tag{2.21.1}$$

If in addition Conjecture 2.20 holds, then the inequality (2.21.1) *is moreover an equality. In case the sign of* $\mathcal{F}$ *is* -1, $\mathrm{Sel}(W)^{\vee}$ *has generic rank* 1 *over* $\mathbf{H}$.

In the sign -1 case, we expect there is an expression for $\mathrm{lgth}_{\mathfrak{p}}\, \mathrm{Sel}(W)^{\vee}_{\mathbf{H}\text{-tors}}$ in terms of the derivative of $\mathcal{L}$ with respect to Θ, a regulator, and some Tamagawa factors. We hope to investigate this in a future paper, cf. also 4.13.

3 Uniform control

3.1. We split the main conjecture into two statements: one bounding the Selmer group from below, and one bounding it from above. Both bounds will be proved indirectly via two sets of *control theorems*, one of which is the topic of this section. The control results in this section relate invariants of Selmer groups over $\mathbf{H}$ to Selmer groups over specializations of (certain extensions of) $\mathbf{H}$ at height 1 primes. Our eventual goal is to obtain a bound of the form

$$\mathrm{lgth}_{\mathfrak{p}}\, \mathrm{Sel}_{\Sigma}(W)^{\vee} \leq \mathrm{lgth}_{\mathfrak{p}}\, Z \tag{3.1.1}$$

for every height 1 prime $\mathfrak{p} \subseteq \mathbf{H}$ (where Z is as defined in 2.15). The basic idea of our method is to reduce to applying the theory of Euler systems over DVRs by considering the corresponding bound at a collection of (possibly non-arithmetic) specializations converging, in an appropriate sense, to $\mathfrak{p}$. We first introduce the notion of *rational primes* of $\mathbf{H}$, which are essentially $\overline{\mathbf{Q}}_p$-points on the rigid space associated to $\mathbf{H}$ (i.e., the "generic fiber" of $\operatorname{Spf}\mathbf{H}$).

3.2. In what follows, for a 2-dimensional local ring A of mixed characteristic $(0, p)$, we denote by $\operatorname{Spec}_1 A$ the subset of $\operatorname{Spec} A$ consisting of height 1 primes not containing (p). Fix a finite extension K of $\mathbf{Q}_p$ with ring of integers $\mathcal{O} = \mathcal{O}_K$ and maximal ideal $\mathfrak{m}$ and set $\Lambda_{\mathcal{O}} = \mathcal{O}[\![\Gamma_D]\!]$. Further assume A is a domain finite and flat over Λ and set $A_{\mathcal{O}} = A \otimes_\Lambda \Lambda_{\mathcal{O}}$. Let $X_A(K) = \operatorname{Hom}_{\mathcal{O}}(A_{\mathcal{O}}, \mathcal{O})$. By associating a homomorphism with its kernel, we may (and do) view $X_A(K)$ as the subset of $\operatorname{Spec}_1 A_{\mathcal{O}}$ consisting of primes with residue field K. Consequently, we refer to elements of $X_A(K)$ as *K-rational primes* of A. (This is slightly abusive terminology, as elements of $X_A(K)$ correspond to primes of $A_{\mathcal{O}}$, not primes of A.) If M is a finitely-generated A-module, set $M_{\mathcal{O}} = M \otimes_A A_{\mathcal{O}}$. Also, define $W_{\mathcal{O}} = (T^*_{\mathcal{O}})^\vee$ and $W^*_{\mathcal{O}} = (T_{\mathcal{O}})^\vee$. Note that, as $\mathcal{O}$ is faithfully flat over $\mathbf{Z}_p$, we have that $\operatorname{Sel}(T_{\mathcal{O}}) = \operatorname{Sel}(T)_{\mathcal{O}}$, $\operatorname{Sel}(W_{\mathcal{O}})^\vee = (\operatorname{Sel}(W)^\vee)_{\mathcal{O}}$, and similarly for other Selmer and Galois cohomology groups.

3.3. Suppose given a flat extension $B \to A$ of rings satisfying the hypotheses imposed above. For a height 1 prime $\mathfrak{p} \subseteq B$, we say that $\mathfrak{p}$ is *unramified* in A if $\mathfrak{p}A_{\mathfrak{s}} = \mathfrak{s}A_{\mathfrak{s}}$ for every prime $\mathfrak{s} \subseteq A$ lying over $\mathfrak{p}$. If $\mathfrak{p} \subseteq B$ is unramified in A, then $\mathfrak{p} = \bigcap_{\mathfrak{s}|\mathfrak{p}} \mathfrak{s}$ by the theory of primary decomposition. Note that every $\mathfrak{s} \in \operatorname{Spec}_1 A$ is unramified in $A_{\mathcal{O}}$.

If a prime $\mathfrak{s} \in \operatorname{Spec}_1 A_{\mathcal{O}}$ is K-rational, then $\mathfrak{s}A_{\mathcal{O}_{K'}}$ is a K'-rational prime for all finite extensions K'/K; in this way we identify $X_A(K)$ with a subset of $X_A(K')$. Set $X_A = \bigcup_K X_A(K)$ and call elements of this set *rational primes* of A. If $\mathfrak{s}$ is a rational prime of A and the field of rationality has not been specified, then we denote by $\mathcal{O}(\mathfrak{s})$ the ring of integers of some choice of K for which $\mathfrak{s}$ is K-rational. The notion of unramified prime extends in an obvious way to rational primes.

3.4. We define $d : X_A \times X_A \to \mathbf{R}_{\geq 0}$ as follows. Given $\mathfrak{s}_1, \mathfrak{s}_2 \in X_A$, choose $K/\mathbf{Q}_p$ sufficiently large so that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are K-rational and set $d(\mathfrak{s}_1, \mathfrak{s}_2) = p^{-k}$ for $k = \operatorname{lgth}_{\mathcal{O}} A_{\mathcal{O}}/(\mathfrak{s}_1 + \mathfrak{s}_2)$. Then $d(\mathfrak{s}_1, \mathfrak{s}_2)$ is independent of the choice of K and d defines a *metric* on the set X_A. In the topology defined by d, X_A is *complete* and classical points are dense. Moreover, when $A = \Lambda$ and X_Λ is identified with $\{x \in \overline{\mathbf{Q}}_p \mid |x|_p < 1\}$, d becomes the "usual" metric associated with this set. There is a more intrinsic definition of this metric topology on X_A: to the ring A (or more precisely the formal scheme $\operatorname{Spf} A$ topologically of finite type over $\operatorname{Spf}\mathbf{Z}_p$), one can associate a "generic fiber", i.e., a rigid space $(\operatorname{Spf} A)^{\mathrm{rig}}$ having the property that $(\operatorname{Spf} A)^{\mathrm{rig}}(K) = \operatorname{Hom}_{\mathbf{Z}_p}(A, K)$ for extensions $K/\mathbf{Q}_p$. One can then check that the canonical topology on $(\operatorname{Spf} A)^{\mathrm{rig}}(\overline{\mathbf{Q}}_p)$ is the same as that defined by the metric d.

By [5, Remark 3.5.2], there are only finitely many height 1 primes of Λ ramified in A. Consequently, all rational primes of Λ sufficiently close but not equal to a given prime are unramified.

3.5. We first note that the inequality (3.1.1) holds for a height 1 prime $\mathfrak{p} \subseteq \mathbf{H}$ not containing (p) if and only if the inequality

$$\operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_{\Sigma}(W_{\mathcal{O}(\mathfrak{s})})^{\vee} \leq \operatorname{lgth}_{\mathfrak{s}} Z_{\mathcal{O}(\mathfrak{s})}$$

holds for the rational primes $\mathfrak{s} \in X_{\mathbf{H}}$ lying over $\mathfrak{p}$. If we choose $\mathfrak{s} \in X_{\mathbf{H}}$ at which the left-hand side of this inequality is positive, then the inequality holds by Lemma 4.9 if we can show that

$$\operatorname{lgth}_{\mathcal{O}(\mathfrak{t})} \operatorname{Sel}_{\Sigma}(W_{\mathcal{O}(\mathfrak{t})})^{\vee}/\mathfrak{t} \operatorname{Sel}_{\Sigma}(W_{\mathcal{O}(\mathfrak{t})})^{\vee} \leq \operatorname{lgth}_{\mathcal{O}(\mathfrak{t})} Z_{\mathcal{O}(\mathfrak{t})}/\mathfrak{t}Z_{\mathcal{O}(\mathfrak{t})} + O(1)$$

for a sequence of rational primes $\mathfrak{t} \in X_{\mathbf{H}}$ converging to $\mathfrak{s}$ in the topology described above. Noting that the left-hand side of this inequality is equal to $\operatorname{lgth}_{\mathcal{O}(\mathfrak{t})} \operatorname{Sel}_{\Sigma}(W_{\mathcal{O}(\mathfrak{t})})[\mathfrak{t}]$, we break the inequality into three steps:

$$\begin{aligned} \text{(3.5.1)} \quad \operatorname{lgth}_{\mathcal{O}(\mathfrak{t})} \operatorname{Sel}_{\Sigma}(W_{\mathcal{O}(\mathfrak{t})})[\mathfrak{t}] &\leq \operatorname{lgth}_{\mathcal{O}(\mathfrak{t})} \operatorname{Sel}_{\Sigma}(W_{\mathcal{O}(\mathfrak{t})}[\mathfrak{t}]) + O(1) \leq \\ &\leq \operatorname{lgth}_{\mathcal{O}(\mathfrak{t})} Z_{\mathfrak{t}} + O(1) \leq \operatorname{lgth}_{\mathcal{O}(\mathfrak{t})} Z_{\mathcal{O}(\mathfrak{t})}/\mathfrak{t}Z_{\mathcal{O}(\mathfrak{t})} + O(1), \end{aligned}$$

where we define $Z_{\mathfrak{t}}$ to be $\mathcal{O}/\{\phi(z_{\mathfrak{t}}) \mid \phi \in \operatorname{Hom}_{\mathcal{O}}(\mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^*_{\mathcal{O}}/\mathfrak{t}T^*_{\mathcal{O}}), \mathcal{O})\}$ for $\mathcal{O} = \mathcal{O}(\mathfrak{t})$ and where $z_{\mathfrak{t}}$ is the image of $z(1)$. Thus, in the notation of Rubin [19], we have $Z_{\mathfrak{t}} = \operatorname{ind}_{\mathcal{O}(\mathfrak{t})}(z_{\mathfrak{t}})$. (Note that $Z_{\mathfrak{t}}$ behaves well with resepct to change in $\mathcal{O}$.) We do not concern ourselves in this section with the second inequality, which we deduce in §4 from the general Euler system theory over the DVR $\mathcal{O}(\mathfrak{t})$. The first and third inequalities are the subject of our control results. For the first, we instead investigate the nominally stronger statement that

$$\bigoplus_{\mathfrak{t}|\mathfrak{q}} \operatorname{Sel}_{\Sigma}(W_{\mathcal{O}(\mathfrak{t})}[\mathfrak{t}]) \longrightarrow \bigoplus_{\mathfrak{t}|\mathfrak{q}} \operatorname{Sel}_{\Sigma}(W_{\mathcal{O}(\mathfrak{t})})[\mathfrak{t}]$$

has cokernel of length bounded independently of $\mathfrak{q}$ for any rational prime $\mathfrak{q} \in X_{\Lambda}$ sufficiently close to the rational prime $\mathfrak{p} \in X_{\Lambda}$ lying below $\mathfrak{s}$.

3.6. Our first step is to prove control theorems for Selmer groups for T viewed as Λ-modules. The calculations involved are greatly simplified by the fact that Λ is *regular*, and in particular its height 1 primes and rational primes are all *principal.* To extend the control statements to $\mathbf{H}$, we compare the control error for specializations near a given prime $\mathfrak{p}$ of Λ to the corresponding error for a sum of specializations near the primes of $\mathbf{H}$ lying over $\mathfrak{p}$.

3.7 Proposition. *For almost all rational primes $\mathfrak{q} \in X_{\Lambda}$, the natural homomorphisms*

$$\operatorname{Sel}_{\Sigma}(W_{\mathcal{O}(\mathfrak{q})}[\mathfrak{q}]) \longrightarrow \operatorname{Sel}_{\Sigma}(W_{\mathcal{O}(\mathfrak{q})})[\mathfrak{q}]$$

are injective and have cokernel of length bounded independently of $\mathfrak{q}$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}_\Sigma(W_{\mathcal{O}(\mathfrak{q})}[\mathfrak{q}]) \arrow[r] \arrow[d] & \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, W_{\mathcal{O}(\mathfrak{q})}[\mathfrak{q}]) \arrow[r] \arrow[d, "\alpha"'] & \bigoplus_{v\in\Sigma} \mathrm{H}^1(\mathbf{Q}_v, W_{\mathcal{O}(\mathfrak{q})}[\mathfrak{q}]) \arrow[d, "\beta"'] \\
0 \arrow[r] & \mathrm{Sel}_\Sigma(W_{\mathcal{O}(\mathfrak{q})})[\mathfrak{q}] \arrow[r] & \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, W_{\mathcal{O}(\mathfrak{q})})[\mathfrak{q}] \arrow[r] & \bigoplus_{v\in\Sigma} \mathrm{H}^1(\mathbf{Q}_v, W_{\mathcal{O}(\mathfrak{q})})[\mathfrak{q}]
\end{tikzcd}$$

arising from the definition of Sel_Σ. By the Snake Lemma, we need to bound the kernel and cokernel of α and the kernel of β. As $\mathfrak{q}$ is principal, we have $\operatorname{coker}\alpha = 0$ and

$$\ker\alpha = W_{\mathcal{O}(\mathfrak{q})}(\mathbf{Q})/\mathfrak{q}W_{\mathcal{O}(\mathfrak{q})}(\mathbf{Q}), \quad \ker\beta = \bigoplus_{v\in\Sigma} W_{\mathcal{O}(\mathfrak{q})}(\mathbf{Q}_v)/\mathfrak{q}W_{\mathcal{O}(\mathfrak{q})}(\mathbf{Q}_v).$$

Thus $\ker\alpha = 0$, as (2.8.2) and [5, Prop. 3.4] imply that $W_{\mathcal{O}(\mathfrak{q})}(\mathbf{Q}) = 0$. If $\mathfrak{q}$ is not one of the finitely many primes in the support of $(W_{\mathcal{O}(\mathfrak{q})}(\mathbf{Q}_v)/W_{\mathcal{O}(\mathfrak{q})}(\mathbf{Q}_v)_{\Lambda\text{-div}})^\vee$, then the length of $\ker\beta$ is bounded by that of the maximal co-pseudo-null quotient of W. ☐

3.8 Lemma. *For any rational prime $\mathfrak{t} \in X_{\mathbf{H}}$, the natural homomorphism*

$$\mathrm{Sel}^\Sigma(W_{\mathcal{O}(\mathfrak{t})}[\mathfrak{t}]) \longrightarrow \mathrm{Sel}^\Sigma(W_{\mathcal{O}(\mathfrak{t})})[\mathfrak{t}]$$

is injective. The same result holds with Sel^Σ *replaced by* Sel *or* Sel_Σ.

Proof. It suffices to show that

$$\mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, W_{\mathcal{O}(\mathfrak{t})}[\mathfrak{t}]) \longrightarrow \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, W_{\mathcal{O}(\mathfrak{t})})[\mathfrak{t}]$$

is injective. The kernel of this homomorphism is $(W_{\mathcal{O}(\mathfrak{t})}/W_{\mathcal{O}(\mathfrak{t})}[\mathfrak{t}])(\mathbf{Q})$, so the lemma again follows from [5, Prop. 3.4]. ☐

3.9 Lemma. *Fix a rational prime $\mathfrak{p} \in X_\Lambda$ which is unramified in $\mathbf{H}$. If $\mathfrak{q} \in X_\Lambda$ and $\mathfrak{t}, \mathfrak{t}' \in X_{\mathbf{H}}$ are distinct primes lying over $\mathfrak{q}$, then $d(\mathfrak{t}, \mathfrak{t}')$ is bounded independently of $\mathfrak{q}$ sufficiently close to $\mathfrak{p}$.*

Proof. For a suitable choice of $g \in \Lambda$, $g \notin \mathfrak{p}$, the homomorphism $\Lambda_g \hookrightarrow \mathbf{H}_g$ is étale, so we may assume that $\mathbf{H}_g$ has the form $\Lambda_g[t]_h/(f)$ for some monic $f \in \Lambda[t]$ and $h \in \Lambda_g[t]$ such that f' is invertible in $\Lambda_g[t]_h$. Any $\mathfrak{q} \in X_\Lambda$ sufficiently close to $\mathfrak{p}$ also has $g \notin \mathfrak{q}$, so we may view $\mathfrak{q}$ as an element of X_{Λ_g}. As $\mathfrak{q}$ is rational for some K, we may write $\mathfrak{q} = (X - q)$ with $q \in \mathcal{O}_K$. Say $f = \sum_i \lambda_i t^i$ with $\lambda_i \in \Lambda_g$. Then the elements of X_A lying over $\mathfrak{q}$ correspond to the roots of $f_q = \sum \lambda_i(q) t^i$. The lemma thus follows from the fact that the roots of f_q vary continuously in q. ☐

3.10 Lemma. *Suppose M is a finitely-generated $\mathbf{H}$-module and fix a rational prime $\mathfrak{p} \in X_\Lambda$. For each $\mathfrak{q} \in X_\Lambda$, fix a choice of $\mathcal{O} = \mathcal{O}_K$ such that all $\mathfrak{t} \in X_{\mathbf{H}}$ lying over $\mathfrak{q}$ are K-rational and let*

$$\beta : M_{\mathcal{O}}/\mathfrak{q}M_{\mathcal{O}} \longrightarrow \bigoplus_{\mathfrak{t}|\mathfrak{q}} M_{\mathcal{O}}/\mathfrak{t}M_{\mathcal{O}} \tag{3.10.1}$$

be the natural homomorphism.

- *If $\mathfrak{p}$ is unramified in $\mathbf{H}$, then* $\operatorname{lgth}_{\mathcal{O}} \operatorname{coker} \beta$ *is bounded independently of* $\mathfrak{q}$ *sufficiently close to $\mathfrak{p}$.*
- *If M is free and $\mathfrak{q}$ is unramified in $\mathbf{H}$, then β is injective.*
- *If M is $\mathbf{H}$-torsion, then* $|\operatorname{lgth}_{\mathcal{O}} \ker \beta - \operatorname{lgth}_{\mathcal{O}} \operatorname{coker} \beta|$ *is bounded for* $\mathfrak{q}$ *sufficiently close to but not equal to $\mathfrak{p}$.*

Note that the lengths being measured in this lemma are independent of the choice of $\mathcal{O}$. If $\mathfrak{p}$ is ramified in $\mathbf{H}$ and belongs to the support of M, then it is not hard to see that the length of the cokernel of (3.10.1) is unbounded as $\mathfrak{q}$ approaches $\mathfrak{p}$; this is the source of our difficulty in dealing with ramified primes.

Proof. Let $\mathfrak{t}_1, \ldots, \mathfrak{t}_k$ be the rational primes of $\mathbf{H}$ lying over $\mathfrak{q}$, so $k \leq \operatorname{rk}_\Lambda \mathbf{H}$. If $k = 1$, then there is nothing to prove, so assume $k > 1$. Note that the annihilator of $\operatorname{coker} \beta$ in $\mathbf{H}_{\mathcal{O}}$ contains $\prod_{i<j}(\mathfrak{t}_i + \mathfrak{t}_j)$. By Lemma 3.9, each $\mathfrak{t}_i + \mathfrak{t}_j$ contains a bounded power $\mathfrak{m}_{\mathcal{O}}^r$ of the maximal ideal of $\mathbf{H}_{\mathcal{O}}$. Thus, $\operatorname{lgth}_{\mathcal{O}} \operatorname{coker} \beta$ can be bounded in terms of k, r, and the number of generators of M as $\mathbf{H}$-module, all of which are bounded independently of $\mathfrak{q}$ sufficiently close to $\mathfrak{p}$.

If M is free and $\mathfrak{q}$ is unramified in $\mathbf{H}$, then the kernel of the natural homomorphism $M_{\mathcal{O}} \to \bigoplus_{\mathfrak{t}|\mathfrak{q}} M_{\mathcal{O}}/\mathfrak{t}M_{\mathcal{O}}$ is

$$\bigcap_{\mathfrak{t}|\mathfrak{q}} (\mathfrak{t}M_{\mathcal{O}}) = \Big(\bigcap_{\mathfrak{t}|\mathfrak{q}} \mathfrak{t}\Big) M_{\mathcal{O}} = \mathfrak{q}M_{\mathcal{O}},$$

where the last equality follows from primary decomposition of $\mathfrak{q}$ in $\mathbf{H}_{\mathcal{O}}$.

We now turn our attention to the last statement. Suppose given a set $\mathcal{S}$ of finitely-generated torsion $\mathbf{H}$-modules for which the statement is true. The statement is then also true for elements of $\mathcal{S}_{\mathcal{O}} = \{S_{\mathcal{O}} \mid S \in \mathcal{S}\}$ and for any module M such that $M_{\mathcal{O}}$ is pseudo-isomorphic to a direct sum of elements of $\mathcal{S}_{\mathcal{O}}$. It therefore suffices to prove the statement for modules of the form $\mathbf{H}_{\mathcal{O}}/\mathfrak{s}^r$ for a rational prime $\mathfrak{s} \in X_{\mathbf{H}}$. By prime avoidance, we may choose an element $s \in \mathfrak{s}$ such that s does not belong to $\mathfrak{s}^2$ or to any other prime of $\mathbf{H}_{\mathcal{O}}$ lying over $\mathfrak{p}$. The natural (surjective) homomorphism

$$\mathbf{H}_{\mathcal{O}}/(s^r) \longrightarrow \mathbf{H}_{\mathcal{O}}/\mathfrak{s}^r$$

thus has kernel supported away from all primes lying over $\mathfrak{p}$, so we may assume that $M_{\mathcal{O}} = \mathbf{H}_{\mathcal{O}}/(s^r)$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}_{\mathcal{O}}/\mathfrak{q}\mathbf{H}_{\mathcal{O}} \arrow[r] \arrow[d, "s^r"'] & \bigoplus_{\mathfrak{t}|\mathfrak{q}} \mathbf{H}_{\mathcal{O}}/\mathfrak{t} \arrow[r] \arrow[d, "s^r"'] & X \arrow[r] \arrow[d, "s^r"'] & 0 \\
0 \arrow[r] & \mathbf{H}_{\mathcal{O}}/\mathfrak{q}\mathbf{H}_{\mathcal{O}} \arrow[r] & \bigoplus_{\mathfrak{t}|\mathfrak{q}} \mathbf{H}_{\mathcal{O}}/\mathfrak{t} \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

where X is defined to make the rows exact. All $\mathfrak{q}$ sufficiently close to but not equal to $\mathfrak{p}$ are unramified, so $\mathfrak{q}\mathbf{H}_{\mathcal{O}} = \bigcap_{\mathfrak{t}|\mathfrak{q}} \mathfrak{t}$ and, by our assumption on s, the first two vertical arrows are injective. Thus, by the Snake Lemma, the kernel and cokernel of $M_{\mathcal{O}}/\mathfrak{q}M_{\mathcal{O}} \to \bigoplus_{\mathfrak{t}|\mathfrak{q}} M_{\mathcal{O}}/\mathfrak{t}M_{\mathcal{O}}$ are equal to the kernel and cokernel of $s^r : X \to X$ and thus have the same length. $\square$

3.11. Consider the commutative diagram
(3.11.1)

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{\mathfrak{t}|\mathfrak{q}} \mathrm{Sel}_{\Sigma}(W_{\mathcal{O}}[\mathfrak{t}]) \arrow[r] \arrow[d, "\alpha"'] & \bigoplus_{\mathfrak{t}|\mathfrak{q}} \mathrm{Sel}_{\Sigma}(W_{\mathcal{O}})[\mathfrak{t}] \arrow[r] \arrow[d, "\beta"'] & A(\mathfrak{q}) \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \mathrm{Sel}_{\Sigma}(W_{\mathcal{O}}[\mathfrak{q}]) \arrow[r] & \mathrm{Sel}_{\Sigma}(W_{\mathcal{O}})[\mathfrak{q}] \arrow[r] & B(\mathfrak{q}) \arrow[r] & 0
\end{tikzcd}$$

where $B(\mathfrak{q})$ and $A(\mathfrak{q})$ are defined to make the rows exact. Note that Proposition 3.7 implies that $\mathrm{lgth}_{\mathcal{O}} B(\mathfrak{q})$ is bounded independently of $\mathfrak{q}$ for almost all $\mathfrak{q}$. We are interested in studying the $\mathcal{O}$-length of $A(\mathfrak{q})$; in particular, we show below that, assuming $\mathrm{Sel}_{\Sigma}(W_{\mathcal{O}})^{\vee}$ is a torsion $\mathbf{H}$-module, $\mathrm{lgth}_{\mathcal{O}} A(\mathfrak{q})$ is bounded for $\mathfrak{q}$ sufficiently close to $\mathfrak{p}$. To do so, we show that the lengths of $\ker\beta$ and $\operatorname{coker}\alpha$ are bounded.

3.12 Proposition. *For a fixed rational prime* $\mathfrak{p} \in X_{\Lambda}$*,* $\mathrm{lgth}_{\mathcal{O}(\mathfrak{q})} A(\mathfrak{q})$ *is of order bounded independently of* $\mathfrak{q}$ *sufficiently close to* $\mathfrak{p}$*.*

Proof. Given $\mathfrak{q}$, choose $\mathcal{O} = \mathcal{O}(\mathfrak{q})$ large enough to contain all $\mathcal{O}(\mathfrak{t})$ for $\mathfrak{t} \mid \mathfrak{q}$. By Lemma 3.10, the natural map $\eta(\mathfrak{q}) : T_{\mathcal{O}}/\mathfrak{q}T_{\mathcal{O}} \to \bigoplus_{\mathfrak{t}|\mathfrak{q}} T_{\mathcal{O}}/\mathfrak{t}T_{\mathcal{O}}$ is injective and $C(\mathfrak{q}) = \operatorname{coker}\eta(\mathfrak{q})$ is finite. In particular, $T_{\mathcal{O}}/\mathfrak{q}T_{\mathcal{O}}$ and $\bigoplus_{\mathfrak{t}|\mathfrak{q}} T_{\mathcal{O}}/\mathfrak{t}T_{\mathcal{O}}$ are finite free $\mathcal{O}$-modules of the same rank and $\eta(\mathfrak{q})$ is an *isogeny* of $\mathcal{O}$-modules. We have the following change of Selmer group formula, due to Perrin-Riou [15]:

$$\frac{\left|\bigoplus_{\mathfrak{t}|\mathfrak{q}} \mathrm{Sel}_{\Sigma}(W_{\mathcal{O}}[\mathfrak{t}])\right|}{|\mathrm{Sel}_{\Sigma}(W_{\mathcal{O}}[\mathfrak{q}])|} = \frac{|C(\mathfrak{q})(\mathbf{R})| \cdot |\,\mathrm{F}^{-} C(\mathfrak{q})|}{|C(\mathfrak{q})|},$$

where the filtration on $C(\mathfrak{q})$ is inherited from that on T in the obvious way. In particular, this formula implies $\mathrm{lgth}_{\mathcal{O}} \bigoplus_{\mathfrak{t}|\mathfrak{q}} \mathrm{Sel}_{\Sigma}(W_{\mathcal{O}}[\mathfrak{t}]) \leq \mathrm{lgth}_{\mathcal{O}} \mathrm{Sel}_{\Sigma}(W_{\mathcal{O}}[\mathfrak{q}])$, so $\mathrm{lgth}_{\mathcal{O}} \ker\alpha \leq \mathrm{lgth}_{\mathcal{O}} \operatorname{coker}\alpha$.

Applying the Snake Lemma to (3.11.1) we get the equality

$$\begin{aligned}&\operatorname{lgth}_{\mathcal{O}} \ker\alpha - \operatorname{lgth}_{\mathcal{O}} \ker\beta + \operatorname{lgth}_{\mathcal{O}} \ker\gamma - \\ &\qquad - \operatorname{lgth}_{\mathcal{O}} \operatorname{coker}\alpha + \operatorname{lgth}_{\mathcal{O}} \operatorname{coker}\beta - \operatorname{lgth}_{\mathcal{O}} \operatorname{coker}\gamma = 0.\end{aligned}$$

By Lemma 3.10, $\operatorname{lgth}_{\mathcal{O}} \ker\beta = \operatorname{lgth}_{\mathcal{O}} \operatorname{coker}\beta$. The fact that

$$\operatorname{lgth}_{\mathcal{O}} \ker\gamma - \operatorname{lgth}_{\mathcal{O}} \operatorname{coker}\gamma = \operatorname{lgth}_{\mathcal{O}} A(\mathfrak{q}) + O(1)$$

together with the bound $\operatorname{lgth}_{\mathcal{O}} \ker\alpha \leq \operatorname{lgth}_{\mathcal{O}} \operatorname{coker}\alpha$ from above thus gives $\operatorname{lgth}_{\mathcal{O}} A(\mathfrak{q}) \leq O(1)$, as desired. □

3.13. For $\mathcal{O}$ the integer ring of a finite extension $K/\mathbf{Q}_p$, recall (cf. 2.14) that we have defined $Z_{\mathcal{O}} = \operatorname{Sel}^{\Sigma}(T^*_{\mathcal{O}})/\langle z(1)_{\Theta}\rangle$ and that we defined (in 3.5) $Z_{\mathfrak{t}} = \operatorname{ind}_{\mathcal{O}(\mathfrak{t})} z_{\mathfrak{t}}$. We need the following proposition in order to show that $\operatorname{lgth}_{\mathcal{O}} Z_{\mathfrak{t}} + O(1) \leq \operatorname{lgth}_{\mathcal{O}} Z_{\mathcal{O}}/\mathfrak{t}Z_{\mathcal{O}}$, provided $\mathfrak{t}$ is bounded away from the primes of $\mathbf{H}$ ramified over Λ. It is the statement of this proposition which prevents us from proving the main theorem, 2.21, for primes of $\mathbf{H}$ ramified over Λ.

3.14 Proposition. *Fix a rational prime $\mathfrak{p} \in X_{\Lambda}$ which is unramified in $\mathbf{H}$. For every rational prime $\mathfrak{q} \in X_{\Lambda}$, choose $\mathcal{O} = \mathcal{O}(\mathfrak{q})$ large enough to contain $\mathcal{O}(\mathfrak{t})$ for all $\mathfrak{t} \in X_{\mathbf{H}}$ with $\mathfrak{t} \mid \mathfrak{q}$. The cokernels of the natural homomorphsims*

$$\bigoplus_{\mathfrak{t}\mid\mathfrak{q}} \operatorname{Sel}^{\Sigma}(T^*_{\mathcal{O}})/\mathfrak{t}\operatorname{Sel}^{\Sigma}(T^*_{\mathcal{O}}) \longrightarrow \bigoplus_{\mathfrak{t}\mid\mathfrak{q}} \mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^*_{\mathcal{O}}/\mathfrak{t}T^*_{\mathcal{O}})$$

and

$$\bigoplus_{\mathfrak{t}\mid\mathfrak{q}} Z_{\mathcal{O}}/\mathfrak{t}Z_{\mathcal{O}} \longrightarrow \bigoplus_{\mathfrak{t}\mid\mathfrak{q}} \mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^*_{\mathcal{O}}/\mathfrak{t}T^*_{\mathcal{O}})/\langle z_{\mathfrak{t}}\rangle$$

have finite length bounded independently of $\mathfrak{q}$ sufficiently close to $\mathfrak{p}$.

Proof. First note that the second statement (concerning $Z_{\mathcal{O}}$) follows from the first (concerning $\operatorname{Sel}^{\Sigma}$). Recalling that $\operatorname{Sel}^{\Sigma}(T^*_{\mathcal{O}}) = \mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^*_{\mathcal{O}})$, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Sel}^{\Sigma}(T^*_{\mathcal{O}})/\mathfrak{q}\operatorname{Sel}^{\Sigma}(T^*_{\mathcal{O}}) \arrow[r] \arrow[d] & \mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^*_{\mathcal{O}}/\mathfrak{q}T^*_{\mathcal{O}}) \arrow[r] \arrow[d] & \mathrm{H}^2(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^*_{\mathcal{O}})[\mathfrak{q}] \arrow[r] \arrow[d] & 0 \\
\bigoplus_{\mathfrak{t}\mid\mathfrak{q}} \operatorname{Sel}^{\Sigma}(T^*_{\mathcal{O}})/\mathfrak{t}\operatorname{Sel}^{\Sigma}(T^*_{\mathcal{O}}) \arrow[r] & \bigoplus_{\mathfrak{t}\mid\mathfrak{q}} \mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^*_{\mathcal{O}}/\mathfrak{t}T^*_{\mathcal{O}}) \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

where X is defined to make the rows exact. As $\mathrm{H}^2(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^*_{\mathcal{O}})[\mathfrak{q}]$ is easily seen to be bounded, it suffices to bound the cokernel of the middle vertical arrow. This cokernel is a submodule of $\mathrm{H}^1\big(\mathbf{Q}_{\Sigma}/\mathbf{Q}, C(\mathfrak{q})\big)$, where $C(\mathfrak{q})$ is the cokernel of the natural injection $T^*_{\mathcal{O}}/\mathfrak{q}T^*_{\mathcal{O}} \to \bigoplus_{\mathfrak{t}\mid\mathfrak{q}} T^*_{\mathcal{O}}/\mathfrak{t}T^*_{\mathcal{O}}$ (cf. Lemma 3.10). By the global Euler-Poincaré characteristic formula, we have

$$\operatorname{lgth}_{\mathcal{O}} \mathrm{H}^1\big(\mathbf{Q}_{\Sigma}/\mathbf{Q}, C(\mathfrak{q})\big) = \operatorname{lgth}_{\mathcal{O}} \mathrm{H}^2\big(\mathbf{Q}_{\Sigma}/\mathbf{Q}, C(\mathfrak{q})\big) - \operatorname{lgth}_{\mathcal{O}} C(\mathfrak{q})(\mathbf{R}) + \operatorname{lgth}_{\mathcal{O}} C(\mathfrak{q})$$

Assuming $\mathfrak{p}$ is unramified, the length of $C(\mathfrak{q})$ is bounded, so it suffices to find a bound for the number of generators of $\mathrm{H}^2\big(\mathbf{Q}_\Sigma/\mathbf{Q}, C(\mathfrak{q})\big)$ as $\mathbf{H}_\mathcal{O}/\mathfrak{q}$-module. The rows in the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^*_\mathcal{O}) \arrow[r] \arrow[d, "\Delta"'] & \mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^*_\mathcal{O}/\mathfrak{q}T^*_\mathcal{O}) \arrow[r] \arrow[d] & 0 \\
\bigoplus_\mathfrak{t} \mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^*_\mathcal{O}) \arrow[r] & \bigoplus_\mathfrak{t} \mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^*_\mathcal{O}/\mathfrak{t}T^*_\mathcal{O}) \arrow[r] & 0
\end{tikzcd}$$

where Δ is the diagonal, are exact, as $\mathrm{Gal}(\mathbf{Q}_\Sigma/\mathbf{Q})$ has p-cohomological dimension 2 (recall $p \neq 2$), so there is an $\mathbf{H}_\mathcal{O}$-linear surjection $\bigoplus_\mathfrak{t} \mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^*_\mathcal{O}) \twoheadrightarrow \mathrm{H}^2\big(\mathbf{Q}_\Sigma/\mathbf{Q}, C(\mathfrak{q})\big)$. The result then follows from the fact that $\mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^*_\mathcal{O})$ is a finitely-generated $\mathbf{H}_\mathcal{O}$-module. $\square$

3.15 Corollary. *Assume Conjecture 4.5 if ρ is not self-dual. For $\mathfrak{p} \in X_\Lambda$ unramified in $\mathbf{H}$,*

$$\sum_{\mathfrak{t}|\mathfrak{q}} (\mathrm{lgth}_\mathcal{O} Z_\mathfrak{t} - \mathrm{lgth}_\mathcal{O} Z_\mathcal{O}/\mathfrak{t}Z_\mathcal{O})$$

is bounded independently of $\mathfrak{q}$ sufficiently close to $\mathfrak{p}$.

Proof. By Proposition 4.11 below (the proof of which does not depend on this corollary), $\mathrm{Sel}^\Sigma(T^*_\mathcal{O})$ is torsion-free of rank 1 over $\mathbf{H}_\mathcal{O}$. Therefore, Proposition 3.14 implies that $\mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, T^*_\mathcal{O}/\mathfrak{t}T^*_\mathcal{O})$ is of rank 1 over $\mathcal{O}$ with torsion submodule of bounded length. This implies that

$$\mathrm{lgth}_\mathcal{O} \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, T^*_\mathcal{O}/\mathfrak{t}T^*_\mathcal{O})/\langle z_\mathfrak{t}\rangle - \mathrm{lgth}_\mathcal{O} Z_\mathfrak{t}$$

is likewise bounded, so the corollary follows from the second bound of 3.14. $\square$

4 Euler system bound

4.1. In order to glean information about the $\mathbf{H}$-module structure of the Selmer groups associated to W using Kato's Euler system, we need to know that this Euler system remains non-trivial when restricted to $\Lambda_\mathbf{H}/\Theta$. When the sign of the Hida family is not -1, then the non-vanishing of the Euler system follows from a conjecture regarding the non-vanishing of L-values of modular forms. When the sign of the Hida family is -1, the situation is somewhat more complicated. There is no known way to relate non-vanishing of the Euler system to the non-vanishing of classical L-functions or their derivatives, but the main results of [1] allow one to deduce the non-vanishing of this Euler system from the non-vanishing of the derivative of $\mathcal{L}$ and a weak Leopoldt conjecture for W.

4.2 Conjecture. *If the sign of $\mathcal{F}$ is not -1, then all but finitely many of the L-values $L(f_\kappa, w(\kappa)/2)$ are non-zero as κ ranges over all arithmetic characters of $\mathbf{H}$ such that $w(\kappa) \equiv k \pmod{p-1}$ and $\psi_\kappa = 1$.*

We only use the following corollary to this conjecture, which would follow from the existence of *one* non-vanishing specialization $L(f_\kappa, w(\kappa)/2)$ satisfying the hypotheses of Conjecture 4.3.

4.3 Conjecture. *Under the natural restriction map, the image* $\mathrm{loc}_{\mathrm{s}}\, z(1)_\Theta \in \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^*)$ *of* $\mathrm{loc}_{\mathrm{s}}\, z(1) \in \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^{\mathrm{no},*})$ *is non-trivial.*

When the sign of $\mathcal{F}$ is -1 the L-values $L(f_\kappa, w(\kappa)/2)$ are forced to vanish for infinitely many choices of κ. Thus $\mathrm{loc}\, z(1)_\Theta \in \mathrm{H}^1(\mathbf{Q}_p, T^*)$ lies in the local condition $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, T^*)$ and cannot be related in any obvious way to L-values which are expected to be non-vanishing. We do, however, expect this class to be non-trivial.

4.4 Conjecture. *If the sign of* $\mathcal{F}$ *is* -1*, then* $\mathrm{loc}\, z(1)_\Theta \in \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, T^*)$ *does not lie in the torsion submodule of* $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, T^*)$*.*

4.5. In the case that ρ is not self-dual, we assume the following conjecture, which allows us to compute the rank of Selmer groups for T^* in terms of those for T. As remarked in 2.21, this conjecture should not be too difficult to prove assuming 4.3 and 4.4 and using the results below. Such a statement in the self-dual case is Theorem 4.15 of [1], though the proof given in *ibid* does not work without self-duality.

Conjecture. *The p-adic height pairing (see [17] or [1, §3] for the definition)*

$$\mathrm{Sel}(T^*) \times \mathrm{Sel}(T) \longrightarrow \mathbf{K}$$

is non-degenerate.

4.6 Proposition. *There are canonical* 5*-term exact sequences*

$$(4.6.1) \quad 0 \longrightarrow \mathrm{Sel}(T^*) \longrightarrow \mathrm{Sel}^{\Sigma}(T^*) \longrightarrow \bigoplus_{v \in \Sigma} \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_v, T^*) \longrightarrow \\ \longrightarrow \mathrm{Sel}(W)^\vee \longrightarrow \mathrm{Sel}_{\Sigma}(W)^\vee \longrightarrow 0$$

and

$$(4.6.2) \quad 0 \longrightarrow \mathrm{Sel}_{\Sigma}(T^*) \longrightarrow \mathrm{Sel}(T^*) \longrightarrow \bigoplus_{v \in \Sigma} \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, T^*) \longrightarrow \\ \longrightarrow \mathrm{Sel}^{\Sigma}(W)^\vee \longrightarrow \mathrm{Sel}(W)^\vee \longrightarrow 0$$

Proof. The existence of these sequences is immediate from the definition of the Selmer groups involved (cf. 2.12) and local duality. □

Not that the sum $\bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_v, T^*)$ in (4.6.1) reduces to the single summand $\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^*)$ in view of the fact [1, Lemma 2.9] that that $\mathrm{H}^1(\mathbf{Q}_v, T^*)$ is a torsion $\mathbf{H}$-module for $v \nmid p$.

4.7 Proposition. *There is a natural isomorphism of* Λ*-modules*

$$\mathrm{Sel}^{\Sigma}(T^*) \longrightarrow \mathrm{Hom}_{\Lambda}(\mathrm{Sel}^{\Sigma}(W^*)^{\vee}, \Lambda).$$

In particular, $\mathrm{Sel}^{\Sigma}(T^*)$ *is* $\mathbf{H}$*-torsion-free of rank* $\mathrm{rk}_{\mathbf{H}} \mathrm{Sel}^{\Sigma}(W^*)^{\vee}$ *over* $\mathbf{H}$*. The same statements hold when when* T^* *is replaced by* T *or when* T^**,* W^**,* Λ*, and* $\mathbf{H}$ *are replaced by* $T^{\mathrm{no},*}$*,* $W^{\mathrm{no},*}$*,* $\Lambda[\![\Gamma_C]\!]$*, and* $\Lambda_{\mathbf{H}}$*, respectively.*

Proof. This proposition and its proof are essentially due to Perrin-Riou, cf. [16, Prop. 4.2.3]. We give here the proof for T^*; the arguments for T and $T^{\mathrm{no},*}$ are very similar. For positive integers n, set $\Lambda_n = \Lambda/(X^n)$, $T_n^* = T^* \otimes_{\Lambda} \Lambda_n = T^*/X^n T^*$ and $W_n^* = W^*[X^n]$. Then, for any positive integer k, we have a short exact sequence

$$0 \longrightarrow T_n^*/p^k T_n^* \longrightarrow W_n^* \xrightarrow{p^k} W_n^* \longrightarrow 0.$$

From the $\mathrm{Gal}(\mathbf{Q}_{\Sigma}/\mathbf{Q})$-cohomology of this sequence, we get a surjective homomorphism

$$\mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T_n^*/p^k T_n^*) \longrightarrow \mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, W_n^*)[p^k] \tag{4.7.1}$$

with kernel $W_n^*(\mathbf{Q})/p^k W_n^*(\mathbf{Q})$. This kernel is trivial by [5, Prop. 3.4], so (4.7.1) is in fact an isomorphism. There are natural identifications

$$\varprojlim_{k} \mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, W_n^*)[p^k] = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, W_n^*)^{\vee}, \mathbf{Z}_p),$$

and

$$\mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, W_n^*)^{\vee}, \mathbf{Z}_p) = \mathrm{Hom}_{\Lambda_n}(\mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, W_n^*)^{\vee}, \Lambda_n),$$

where the first equality follows from the fact that W_n^* is cofinitely-generated over $\mathbf{Z}_p$ and the second is induced by the natural quotient $\Lambda_n \to \mathbf{Z}_p$. Thus, taking the inverse limit of (4.7.1) first in k and then in n gives an isomorphism

$$\mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^*) \cong \mathrm{Hom}_{\Lambda}(\mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, W)^{\vee}, \Lambda).$$

By [1, Lemma 2.10], $\mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^*) = \mathrm{Sel}^{\Sigma}(T^*)$. It therefore remains to show that $\mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, W^*)$ may be replaced by $\mathrm{Sel}^{\Sigma}(W^*)$, for which it suffices to show that the quotient $\mathrm{H}^1(\mathbf{Q}_{\Sigma}/\mathbf{Q}, W^*)/\mathrm{Sel}^{\Sigma}(W^*)$ is $\mathbf{H}$-cotorsion. This quotient injects into $\bigoplus_{v \in \Sigma - \{p\}} \mathrm{H}^1_s(\mathbf{Q}_v, W^*)$, which is $\mathbf{H}$-cotorsion by [1, Lemma 2.9] and local duality.

The statements concerning $\mathbf{H}$-module structure follow from the fact that $\mathbf{H}$ is finite and free over Λ. $\square$

4.8 Lemma. *Fix an integer* a*, a rational prime* $\mathfrak{s} \in X_{\mathbf{H}}$*, and a sequence of rational primes* $\mathfrak{t}_j \in X_{\mathbf{H}}$ *converging to* $\mathfrak{s}$*. Choose* $\mathcal{O}(\mathfrak{t}_j)$ *so that* $\mathcal{O}(\mathfrak{t}_j) \supseteq \mathcal{O}(\mathfrak{s})$ *for all* j*. Then the difference*

$$\mathrm{lgth}_{\mathcal{O}(\mathfrak{t}_j)} \mathbf{H}_{\mathcal{O}(\mathfrak{t}_j)}/(\mathfrak{s}^a + \mathfrak{t}_j) - a \,\mathrm{lgth}_{\mathcal{O}(\mathfrak{t}_j)} \mathbf{H}_{\mathcal{O}(\mathfrak{t}_j)}/(\mathfrak{s} + \mathfrak{t}_j)$$

is bounded as j *varies.*

Proof. In the case that $\mathfrak{s} = (s)$ is principal, then $\mathbf{H}_{\mathcal{O}(\mathfrak{t}_j)}/\big((s)^a + \mathfrak{t}_j\big)$ has a filtration of length a with all quotients isomorphic to $\mathbf{H}_{\mathcal{O}(\mathfrak{t}_j)}/\big((s) + \mathfrak{t}_j\big)$. If $\mathfrak{s}$ is not principal, then, as in the proof of 3.10, replacing $\mathfrak{s}$ by a principal ideal (s), where $s \in \mathfrak{s}$ does not belong to $\mathfrak{s}^2$ or to any prime lying over $\mathfrak{s} \cap \Lambda_{\mathcal{O}(\mathfrak{s})}$, changes both terms in the difference by only a bounded amount. $\square$

4.9 Lemma. *Let M and N be torsion $\mathbf{H}$-modules and $\mathfrak{s} \in X_{\mathbf{H}}$ a rational prime belonging to the support of M. Suppose given a sequence $\mathfrak{t}_j \in X_{\mathbf{H}}$ of rational primes not lying in $\operatorname{Supp} M \cup \operatorname{Supp} N$ which converge to $\mathfrak{s}$. Then*

$$\operatorname{lgth}_{\mathfrak{s}} M_{\mathcal{O}(\mathfrak{s})} \leq \operatorname{lgth}_{\mathfrak{s}} N_{\mathcal{O}(\mathfrak{s})}$$

if and only if the difference

$$\operatorname{lgth}_{\mathcal{O}(\mathfrak{t}_j)} M_{\mathcal{O}(\mathfrak{t}_j)}/\mathfrak{t}_j M_{\mathcal{O}(\mathfrak{t}_j)} - \operatorname{lgth}_{\mathcal{O}(\mathfrak{t}_j)} N_{\mathcal{O}(\mathfrak{t}_j)}/\mathfrak{t}_j N_{\mathcal{O}(\mathfrak{t}_j)}$$

is bounded above.

Proof. As pseudo-null $\mathbf{H}$-modules have finite $\mathbf{Z}_p$-length, replacing M and N by pseudo-isomorphic modules does not change the statement of the lemma. We may therefore assume, for suitable $\mathcal{O} = \mathcal{O}_K$ containing $\mathcal{O}(\mathfrak{s})$, that $M_{\mathcal{O}}$ and $N_{\mathcal{O}}$ are sums of cyclic modules of the form

$$M_{\mathcal{O}} = \bigoplus_{i=1}^{n} \mathbf{H}_{\mathcal{O}}/\mathfrak{r}_i^{a_i} \quad \text{and} \quad N_{\mathcal{O}} = \bigoplus_{i=1}^{n} \mathbf{H}_{\mathcal{O}}/\mathfrak{r}_i^{b_i}$$

with rational primes $\mathfrak{r}_i \in X_{\mathbf{H}}$. We should remark that, with this convention, if M is not pseudo-isomorphic to N, then one or more of the a_i or b_i will be equal to 0.

If $\mathfrak{r}$ is distinct from $\mathfrak{s}$, then, for a fixed integer k, $\operatorname{lgth}_{\mathcal{O}(\mathfrak{t}_j)} \mathbf{H}_{\mathcal{O}(\mathfrak{t}_j)}/(\mathfrak{r}^k + \mathfrak{t}_j)$ is bounded as $\mathfrak{t}_i \to \mathfrak{s}$ (choosing of course $\mathcal{O}(\mathfrak{t}_j)$ to contain $\mathcal{O}(\mathfrak{r})$). We may thus further assume that $\mathfrak{r}_i = \mathfrak{s}$ for all i. Under this assumption, by Lemma 4.8, we have that $\operatorname{lgth}_{\mathcal{O}(\mathfrak{t}_j)} M_{\mathcal{O}(\mathfrak{t}_j)}/\mathfrak{t}_j M_{\mathcal{O}(\mathfrak{t}_j)}$ differs by a bounded amount from

$$-\sum_i a_i \log_p d(\mathfrak{s}, \mathfrak{t}_j),$$

and similarly for N. The lemma thus follows from the fact that

$$\operatorname{lgth}_{\mathcal{O}(\mathfrak{s})} M_{\mathcal{O}(\mathfrak{s})} = \sum_i a_i \quad \text{and} \quad \operatorname{lgth}_{\mathcal{O}(\mathfrak{s})} N_{\mathcal{O}(\mathfrak{s})} = \sum_i b_i,$$

as $\mathfrak{t}_j \to \mathfrak{s}$ means by definition $d(\mathfrak{s}, \mathfrak{t}_j) \to 0$. $\square$

4.10 Proposition. *Assume Conjectures 4.3 and 4.4 and that either ρ is self-dual or Conjecture 4.5 holds.* $\operatorname{Sel}_\Sigma(W)^\vee$ *is a torsion* $\mathbf{H}$*-module and*

$$\operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_\Sigma(W)^\vee \leq \operatorname{lgth}_{\mathfrak{s}} Z$$

for every height 1 *prime* $\mathfrak{s} \subseteq \mathbf{H}$ *not of type* (1.5.1)–(1.5.4)*. If* $\overline{\rho}(G_{\mathbf{Q}}) \supseteq \mathrm{GL}_2(\mathbf{F}_p)$*, then we only need exclude primes of type* (1.5.1)–(1.5.3).

Proof. It remains to prove the second inequality in (3.5.1), which we deduce from the general Euler system theory. Our desired bound follows from [19, Thm. 2.2.2], provided that we can guarantee the existence of certain elements in the images of $\rho_{\mathfrak{t}}$ (denoted τ in [19]) and that we can bound the error terms (denoted $\mathfrak{n}_W$ and $\mathfrak{n}_W^*$ in *ibid*). That $\mathrm{Sel}_\Sigma(W)^\vee$ is a torsion $\mathbf{H}$-module is a by-product of the Euler system argument and the injection in the top row of (3.11.1) (this injectivity is a consequence of Proposition 3.7).

As we are assuming $\mathrm{Im}\,\rho$ contains $\Gamma(\mathfrak{c}) = \mathrm{SL}_2(\Lambda) \cap \big(1 + \mathfrak{c} \cdot \mathrm{M}_2(\Lambda)\big)$ for an ideal $\mathfrak{c} \subseteq \Lambda$ prime to $\mathfrak{s}$, the rational primes $\mathfrak{t}$ sufficiently close to (a given rational prime dividing) $\mathfrak{s}$ are bounded away from the rational primes of $\mathbf{H}$ dividing $\mathfrak{c}$. Thus, in the image of $\rho_{\mathfrak{t}}$, we can find an element of the form

$$\tau_{\mathfrak{t}} = \begin{pmatrix} 1 & x_{\mathfrak{t}} \\ 0 & 1 \end{pmatrix},$$

where $\mathrm{lgth}_{\mathcal{O}(\mathfrak{t})}\, \mathcal{O}(\mathfrak{t})/(x_{\mathfrak{t}})$ is bounded as $\mathfrak{t}$ varies. The bound of [19, Thm. 2.2.2] therefore holds up to bounded error, which is sufficient for our purposes.

It remains to discuss the error terms $\mathfrak{n}_W$ and $\mathfrak{n}_W^*$. By assumption, the representations $\rho_{\mathfrak{t}}$ and $\rho_{\mathfrak{t}}^*$ have open image, at least for a dense set of $\mathfrak{t} \in X_{\mathbf{H}}$. Recalling that $\Omega = \mathbf{Q}(\mu_{p^\infty}, T)$, it suffices for this to bound the torsion in $\mathrm{H}^2(\Omega/\mathbf{Q}, T^*_{\mathcal{O}(\mathfrak{t})}/\mathfrak{t}T^*_{\mathcal{O}(\mathfrak{t})} \oplus T_{\mathcal{O}(\mathfrak{t})}/\mathfrak{t}T_{\mathcal{O}(\mathfrak{t})})$ by the proof of [19, Thm. C.2.2]. We argue as in the proof of Proposition 3.14 to establish that the natural homomorphism

$$\mathrm{H}^2(\Omega/\mathbf{Q}, T^* \oplus T) \otimes_{\mathbf{H}} \mathbf{H}_{\mathcal{O}}/\mathfrak{t} \longrightarrow \mathrm{H}^2(\Omega/\mathbf{Q}, T^*_{\mathcal{O}}/\mathfrak{t}T^*_{\mathcal{O}} \oplus T_{\mathcal{O}}/\mathfrak{t}T_{\mathcal{O}})$$

has bounded cokernel. Note that we are assuming that $\mathfrak{t}$ is bounded away from the ramified primes. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2(\Omega/\mathbf{Q}, T^* \oplus T) \otimes_{\mathbf{H}} \mathbf{H}_{\mathcal{O}}/\mathfrak{q} \arrow[r] \arrow[d] & \mathrm{H}^2(\Omega/\mathbf{Q}, T^*_{\mathcal{O}}/\mathfrak{q}T^*_{\mathcal{O}} \oplus T_{\mathcal{O}}/\mathfrak{q}T_{\mathcal{O}}) \arrow[r] \arrow[d, "\phi"] & 0 \\
\bigoplus_i \mathrm{H}^2(\Omega/\mathbf{Q}, T^* \oplus T) \otimes_{\mathbf{H}} \mathbf{H}_{\mathcal{O}}/\mathfrak{t}_i \arrow[r] & \bigoplus_i \mathrm{H}^2(\Omega/\mathbf{Q}, T^*_{\mathcal{O}}/\mathfrak{t}_iT^*_{\mathcal{O}} \oplus T_{\mathcal{O}}/\mathfrak{t}_iT_{\mathcal{O}}) \arrow[r] & Y
\end{tikzcd}$$

where $\{\mathfrak{t}_i\}$ is the set of primes lying over the prime $\mathfrak{q} \in X_\Lambda$ over which $\mathfrak{t}$ lies and Y is defined to make the bottom row exact. Note that the top row is exact, as $\mathrm{Gal}(\Omega/\mathbf{Q})$ has cohomological dimension 2. As in Proposition 3.14, ϕ has bounded cokernel, so Y is likewise bounded.

Consequently, $\mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^*_{\mathcal{O}(\mathfrak{t})}/\mathfrak{t}T^*_{\mathcal{O}(\mathfrak{t})} \oplus T_{\mathcal{O}(\mathfrak{t})}/\mathfrak{t}T_{\mathcal{O}(\mathfrak{t})})_{\mathbf{Z}_p\text{-tors}}$ differs by only a bounded amount from $\mathrm{H}^2(\Omega/\mathbf{Q}, T^* \oplus T)_{\mathbf{H}\text{-tors}} \otimes \mathbf{H}_{\mathcal{O}(\mathfrak{t})}/\mathfrak{t}$, which has bounded length if $\mathfrak{t}$ is bounded away from the support of $\mathrm{H}^2(\Omega/\mathbf{Q}, T^* \oplus T)_{\mathbf{H}\text{-tors}}$.

If $\overline{\rho}(G_{\mathbf{Q}}) \supseteq \mathrm{SL}_2(\mathbf{F}_p)$, then $\mathrm{H}^1(\rho_{\mathfrak{t}}(G_{\mathbf{Q}}), W^*_{\mathcal{O}}[\mathfrak{t}] \oplus W_{\mathcal{O}}[\mathfrak{t}]) = 0$. Setting $\Omega_{\mathfrak{t}} = \mathbf{Q}(\mu_{p^\infty}, W_{\mathcal{O}}[\mathfrak{t}]) = \mathbf{Q}(\mu_{p^\infty}, W^*_{\mathcal{O}}[\mathfrak{t}])$, one can give an argument similar to [19, Thm. C.2.2(i)] to show that $\mathrm{H}^1(\Omega/\mathbf{Q}, W^*_{\mathcal{O}}[\mathfrak{t}] \oplus W_{\mathcal{O}}[\mathfrak{t}]) = 0$, which of course implies the vanishing of the error terms in the Euler system bound. □

4.11 Proposition. *Assume Conjectures 4.3 and 4.4 and that either* ρ *is self-dual or Conjecture 4.5 holds.* $\mathrm{Sel}^{\Sigma}(T^*)$ *is torsion-free of generic rank* 1 *over* $\mathbf{H}$*. If the sign of* $\mathcal{F}$ *is* -1*, then* $\mathrm{Sel}(W)^{\vee}$ *has generic rank* 1 *over* $\mathbf{H}$*; otherwise,* $\mathrm{Sel}(W)^{\vee}$ *is a torsion* $\mathbf{H}$*-module.*

Proof. We claim that $\mathrm{rk}_{\mathbf{H}}\,\mathrm{Sel}^{\Sigma}(W)^{\vee} \leq 1$. Granting this, Proposition 4.7 implies the bound $\mathrm{rk}_{\mathbf{H}}\,\mathrm{Sel}^{\Sigma}(T)^{\vee} \leq 1$ and self-duality or Conjecture 4.5 then imply that $\mathrm{rk}_{\mathbf{H}}\,\mathrm{Sel}^{\Sigma}(T^*)^{\vee} \leq 1$. The remainder of the proof is a rank analysis using 4.6 and, in particular, the fact that the local terms in the middle of (4.6.1) and (4.6.2) have $\mathbf{H}$-rank 1.

$\mathrm{Sel}_{\Sigma}(W)^{\vee}$ is a torsion $\mathbf{H}$-module by Proposition 4.10. (Note that the proof of this statement does not depend on Corollary 3.15, so we are not giving a circular argument.) Thus, (4.6.1) implies $\mathrm{rk}_{\mathbf{H}}\,\mathrm{Sel}(W)^{\vee} \leq 1$. If $\mathrm{rk}_{\mathbf{H}}\,\mathrm{Sel}(W)^{\vee} = 0$, then by (4.6.2) our claimed bound $\mathrm{rk}_{\mathbf{H}}\,\mathrm{Sel}^{\Sigma}(W)^{\vee} \leq 1$ in this case. In the case that the sign of $\mathcal{F}$ is not -1, Conjecture 4.3 implies that we are in this case. Therefore, let us assume that we are in the sign -1 case and that $\mathrm{rk}_{\mathbf{H}}\,\mathrm{Sel}(W)^{\vee} = 1$. By Conjecture 4.4, we must have that $\mathrm{rk}_{\mathbf{H}}\,\mathrm{Sel}(W)^{\vee} = \mathrm{rk}_{\mathbf{H}}\,\mathrm{Sel}^{\Sigma}(W)^{\vee}$, which again gives our desired bound.

The first statement of the proposition now follows from Conjectures 4.3 and 4.4, as $z(1)_{\Theta}$ provides a non-torsion class in $\mathrm{Sel}^{\Sigma}(T^*)$. The remaining statements now can be proved by an analysis of the ranks of the terms in (4.6.1) and (4.6.2). □

4.12. To complete the "$\leq$" half of Theorem 2.21, it remains to prove that $\mathrm{lgth}_{\mathfrak{s}}\,\mathrm{Sel}(W)^{\vee} = \mathrm{ord}_{\mathfrak{s}}\,\Xi(\mathrm{loc}_{\mathrm{s}}\,z(1)_{\Theta})$. Let $\mathfrak{s} \subseteq \mathbf{H}$ be any height 1 prime not containing (p) and consider the exact sequence

$$0 \longrightarrow Z \longrightarrow \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^*)/\langle \mathrm{loc}_{\mathrm{s}}\,z(1)_{\Theta}\rangle \longrightarrow \mathrm{Sel}(W)^{\vee} \longrightarrow \mathrm{Sel}_{\Sigma}(W)^{\vee} \longrightarrow 0$$

obtained from (4.6.1). After localizing at $\mathfrak{s}$, all terms in the sequence have finite length over $\mathbf{H}_{\mathfrak{s}}$. The length of the two outside terms is the same by Proposition 5.9 and Theorem 4.10, so the two middle terms must also have the same length. Theorem 2.19 then completes the argument.

4.13. We remark here how the p-adic height pairing (cf. 4.5) relates to the sign -1 case of Theorem 2.21. As in [1, 3.3], let $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, T)^{\mathrm{u}} = \mathrm{cor}_{\infty}\big(\mathrm{H}^1(\mathbf{Q}_{\infty,v}, T)\big)$ and let $\mathrm{Sel}(T)^{\mathrm{u}}$ denote the classes in $\mathrm{Sel}(T)$ belonging to $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, T)^{\mathrm{u}}$ locally everywhere. Denote by h^{u}_{∞} the restriction of h_{∞} to $\mathrm{Sel}(T)^{\mathrm{u}} \times \mathrm{Sel}(T^*)^{\mathrm{u}}$. By construction, $\mathrm{Im}\,h^{\mathrm{u}}_{\infty} \subseteq \mathbf{H}$, so (abusing notation) we may view h^{u}_{∞} as a homomorphism $\mathrm{Sel}(T)^{\mathrm{u}} \to \mathrm{Hom}_{\mathbf{H}}(\mathrm{Sel}(T^*)^{\mathrm{u}}, \mathbf{H})$. Set $\mathrm{coker}\,h^{\mathrm{u}}_{\infty} = R^{\mathrm{u}}$. We then conjecture that, in the sign -1 case,

$$\mathrm{ord}_{\mathfrak{s}}(\Theta^{-1}\mathcal{L})_{\Theta} = \mathrm{lgth}_{\mathfrak{s}}\,R^{\mathrm{u}} + \sum_{v} \mathrm{lgth}_{\mathfrak{s}}\,\ell_v(T) + \mathrm{lgth}_{\mathfrak{s}}(\mathrm{Sel}(W)^{\vee})_{\mathrm{tors}}$$

for all height 1 primes $\mathfrak{s} \subseteq \mathbf{H}$, where the $\ell_v(T)$ are the "Tamagawa factors" $\ell_v(T) = \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, T)/\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, T)^{\mathrm{u}}$.

5 Conjectural descent bound

5.1. In this section, we complete the proof of Theorem 2.21, appealing to Conjecture 2.20 and proving a control theorem allowing the passage of information from Selmer groups over $\Lambda_{\mathbf{H}}$ to Selmer groups over $\Lambda_{\mathbf{H}}/\Theta \cong \mathbf{H}$. Though many of the statements in this section are "well-known", at least in more classical contexts (in particular, cf. Ochiai [14, §5], where similar statements are proved), it is nonetheless worthwhile to indicate the arguments specific to our case.

5.2. Our goal in the next few paragraphs is to prepare some preliminary statements which allow us to eventually conclude an inequality of the form

$$\operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_\Sigma(W)^\vee \geq \operatorname{lgth}_{\mathfrak{s}} Z.$$

The proof of this inequality is broken into several steps which we summarize here. First,

$$\operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_\Sigma(W)^\vee \geq \operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_\Sigma(W^{\mathrm{no}})^\vee/\Theta \operatorname{Sel}_\Sigma(W^{\mathrm{no}})^\vee$$

is Proposition 5.4. Next, we show the inequality

$$\begin{aligned}\operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_\Sigma(W^{\mathrm{no}})^\vee/\Theta \operatorname{Sel}_\Sigma(W^{\mathrm{no}})^\vee &\geq \\ &\geq \sum_{\Theta\notin\mathfrak{P}} \mu_{\mathfrak{s}}(\mathfrak{P}) \operatorname{lgth}_{\mathfrak{P}} \operatorname{Sel}_\Sigma(W^{\mathrm{no}})^\vee + \operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_\Sigma(W^{\mathrm{no}})^\vee[\Theta]\end{aligned}$$

in Lemma 5.5, where we write $\mu_{\mathfrak{s}}(\mathfrak{P}) = \operatorname{lgth}_{\mathfrak{s}} \Lambda_{\mathbf{H}}/(\Theta, \mathfrak{P})$ for height 1 primes $\mathfrak{P} \subseteq \Lambda_{\mathbf{H}}$, $\Theta \notin \mathfrak{P}$, and $\mathfrak{s} \subseteq \mathbf{H}$. In §3, we show how the inequality

$$\operatorname{lgth}_{\mathfrak{P}} \operatorname{Sel}_\Sigma(W^{\mathrm{no}})^\vee \geq \operatorname{lgth}_{\mathfrak{P}} Z^{\mathrm{no}}$$

for $\mathfrak{P} \in \operatorname{Spec}_1 \Lambda_{\mathbf{H}}$ is essentially equivalent to Conjecture 2.20. As we show in §3, our desired inequality follows upon combining the above with the bound

$$\operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_\Sigma(W^{\mathrm{no}})^\vee[\Theta] \geq \operatorname{lgth}_{\mathfrak{s}} \mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^{\mathrm{no},*})[\Theta]$$

of Lemma 5.6.

5.3 Lemma. *For any prime $v \nmid p$, the natural homomorphisms*

$$\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_v, W) \to \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_v, W^{\mathrm{no}})$$

are injective. Similarly, the natural homomorphism

$$\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, W) \to \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, W^{\mathrm{no}})$$

is injective.

Proof. This lemma is essentially [14, Prop. 5.2(d)]. If $v \nmid p$, then $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, W) = 0$, so the kernel for $\mathrm{H}^1_{\mathrm{s}}$ is isomorphic to $W^{\mathrm{no}}(\mathbf{Q}_v)/\Theta W^{\mathrm{no}}(\mathbf{Q}_v)$. Denoting by I_v the inertia subgroup of G_{K_v}, the Pontryagin dual of this kernel is thus isomorphic to

$$\big((W^\vee)_{I_v} \otimes_\Lambda \Lambda_{\mathbf{H}}(\chi^{\mathrm{univ},-1}\theta^{-1})\big)_{G_{K_v}/I_v}[\Theta],$$

which is trivial. The argument for $v = p$ is similar. ☐

5.4 Proposition (cf. [14, Prop. 5.2])**.** *The natural homomorphism*

$$\mathrm{Sel}^\Sigma(W) \longrightarrow \mathrm{Sel}^\Sigma(W^{\mathrm{no}})[\Theta]$$

is an isomorphism. The same result holds with Sel^Σ *replaced by* Sel *or* Sel_Σ*.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}(W) \arrow[r] \arrow[d] & \mathrm{Sel}^\Sigma(W) \arrow[r] \arrow[d] & \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_v, W) \arrow[d] \\
0 \arrow[r] & \mathrm{Sel}(W^{\mathrm{no}})[\Theta] \arrow[r] & \mathrm{Sel}^\Sigma(W^{\mathrm{no}})[\Theta] \arrow[r] & \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_v, W^{\mathrm{no}})[\Theta]
\end{tikzcd}$$

We have that $\mathrm{Sel}^\Sigma(W) = \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, W)$, so the middle vertical arrow is an isomorphism by [5, Prop. 3.4], from which we may immediately conclude injectivity in the case of Sel^Σ and surjectivity in all cases. Lemma 5.3 further gives the statement for Sel.

The same argument using the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}_\Sigma(W) \arrow[r] \arrow[d] & \mathrm{Sel}(W) \arrow[r] \arrow[d] & \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, W) \arrow[d] \\
0 \arrow[r] & \mathrm{Sel}_\Sigma(W^{\mathrm{no}})[\Theta] \arrow[r] & \mathrm{Sel}(W^{\mathrm{no}})[\Theta] \arrow[r] & \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, W^{\mathrm{no}})[\Theta]
\end{tikzcd}$$

gives the statement for Sel_Σ. ☐

5.5 Lemma (cf. Rubin [18, Lemma 6.2(ii)])**.** *Suppose M is a torsion $\Lambda_{\mathbf{H}}$-module such that $M/\Theta M$ is a torsion $\mathbf{H}$-module. Then $M[\Theta]$ is likewise $\mathbf{H}$-torsion and, for any height* 1 *prime $\mathfrak{s} \subseteq \mathbf{H}$,*

$$\mathrm{lgth}_{\mathfrak{s}}\, M/\Theta M = \sum_{\Theta\notin\mathfrak{P}} \mu_{\mathfrak{s}}(\mathfrak{P})\, \mathrm{lgth}_{\mathfrak{P}}\, M + \mathrm{lgth}_{\mathfrak{s}}\, M[\Theta].$$

Proof. Set $R = \mathbf{H}_{\mathfrak{s}}$ and $\Lambda_R = \Lambda_{\mathbf{H}} \hat{\otimes}_{\mathbf{H}} R \cong R[\![X]\!]$. Note that R is a DVR, so Λ_R is a 2-dimensional regular local ring. We abuse notation and still write Θ for the ideal $\Theta\Lambda_R$ (or its generator). Setting $M_R = M \otimes_{\Lambda_{\mathbf{H}}} \Lambda_R$, we may replace M

by M_R in the statement of the lemma, where now we understand $\mathfrak{P}$ to run over height 1 primes of Λ_R not containing Θ. Fix an exact sequence

$$0 \longrightarrow \bigoplus_i \Lambda_R/(f_i) \longrightarrow M_R \longrightarrow N \longrightarrow 0$$

with N a torsion R-module. Using the Snake Lemma and applying $\operatorname{lgth}_{\mathfrak{s}}$,

$$\operatorname{lgth}_{\mathfrak{s}} M[\Theta] + \sum_i \operatorname{lgth}_{\mathfrak{s}} R/(\overline{f}_i) + \operatorname{lgth}_{\mathfrak{s}} N/\Theta N = \\ = \operatorname{lgth}_{\mathfrak{s}} N[\Theta] + \operatorname{lgth}_{\mathfrak{s}} M/\Theta M,$$

where $\overline{f}_i$ denotes the image of f_i in $\Lambda_R/\Theta = R$. The lemma then follows from the fact that $\operatorname{lgth}_{\mathfrak{s}} N/\Theta N = \operatorname{lgth}_{\mathfrak{s}} N[\Theta]$. □

5.6 Lemma. *For every height* 1 *prime* $\mathfrak{s} \subseteq \mathbf{H}$,

$$\operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_\Sigma(W^{\mathrm{no}})^\vee[\Theta] = \operatorname{lgth}_{\mathfrak{s}} \mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^{\mathrm{no},*})[\Theta].$$

Proof. An argument similar to Perrin-Riou [16, 4.1.1] gives an exact sequence

$$0 \longrightarrow \operatorname{Sel}_\Sigma(W^{\mathrm{no}})^\vee \longrightarrow \mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^{\mathrm{no},*}) \longrightarrow \bigoplus_{v\in\Sigma} W^{\mathrm{no}}(\mathbf{Q}_v)^\vee.$$

Taking Θ-torsion thus gives the result in light of the fact (cf. the proof of Lemma 5.3) that $W^{\mathrm{no}}(\mathbf{Q}_v)/\Theta W^{\mathrm{no}}(\mathbf{Q}_v) = 0$. □

5.7 Lemma. *For every height* 1 *prime* $\mathfrak{s} \subseteq \mathbf{H}$,

$$\operatorname{lgth}_{\mathfrak{s}} Z^{\mathrm{no}}/\Theta Z^{\mathrm{no}} + \operatorname{lgth}_{\mathfrak{s}} \mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^{\mathrm{no},*})[\Theta] \geq \operatorname{lgth}_{\mathfrak{s}} Z.$$

Proof. The short exact sequence

$$0 \longrightarrow T^{\mathrm{no},*} \xrightarrow{\Theta} T^{\mathrm{no},*} \longrightarrow T^* \longrightarrow 0,$$

gives rise to the cohomology sequence

$$0 \longrightarrow \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, T^{\mathrm{no},*})/\Theta \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, T^{\mathrm{no},*}) \longrightarrow \\ \longrightarrow \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, T^*) \longrightarrow \mathrm{H}^2(\mathbf{Q}_\Sigma/\mathbf{Q}, T^{\mathrm{no},*})[\Theta] \longrightarrow 0.$$

The statement then follows from the Snake Lemma and the definition of Z and Z^{no}. □

5.8. As outlined in 5.2, we may now apply the above results to deduce a lower bound for the size of Selmer groups from the bound of Conjecture 2.20. In the case of sign 1, this gives one inequality in the main conjecture.

5.9 Proposition. *Assume Conjecture 2.20. For every height* 1 *prime* $\mathfrak{s} \subseteq \mathbf{H}$ *not containing* (p)*, we have the inequality*

$$\operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_{\Sigma}(W)^{\vee} \geq \operatorname{lgth}_{\mathfrak{s}} Z.$$

Proof. We make precise the proof sketched in 5.2. Proposition 5.4 gives the bound

$$\operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_{\Sigma}(W)^{\vee} \geq \operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_{\Sigma}(W^{\mathrm{no}})^{\vee}/\Theta \operatorname{Sel}_{\Sigma}(W^{\mathrm{no}})^{\vee}.$$

Lemma 5.5 gives further that

$$\begin{aligned} (5.9.1) \quad & \operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_{\Sigma}(W^{\mathrm{no}})^{\vee}/\Theta \operatorname{Sel}_{\Sigma}(W^{\mathrm{no}})^{\vee} \geq \\ & \qquad \geq \sum_{\mathfrak{P}} \mu_{\mathfrak{s}}(\mathfrak{P}) \operatorname{lgth}_{\mathfrak{P}} \operatorname{Sel}_{\Sigma}(W^{\mathrm{no}})^{\vee} + \operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_{\Sigma}(W^{\mathrm{no}})^{\vee}[\Theta]. \end{aligned}$$

Under Conjecture 2.20, $\operatorname{Sel}(W^{\mathrm{no}})^{\vee}$ is a torsion $\Lambda_{\mathbf{H}}$-module and, for every height 1 prime $\mathfrak{P} \subseteq \Lambda_{\mathbf{H}}$ not containing (p), $\operatorname{lgth}_{\mathfrak{P}} \operatorname{Sel}(W^{\mathrm{no}})^{\vee} \geq \operatorname{ord}_{\mathfrak{P}} \mathcal{L}$. Using the exact sequence

$$0 \longrightarrow Z^{\mathrm{no}} \longrightarrow \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^{\mathrm{no},*})/\langle \operatorname{loc}_{\mathrm{s}} z(1) \rangle \longrightarrow \operatorname{Sel}(W^{\mathrm{no}})^{\vee} \longrightarrow \operatorname{Sel}_{\Sigma}(W^{\mathrm{no}})^{\vee} \longrightarrow 0$$

arising from (4.6.1), we have that $\operatorname{lgth}_{\mathfrak{P}} \operatorname{Sel}_{\Sigma}(W^{\mathrm{no}})^{\vee} = \operatorname{lgth}_{\mathfrak{P}} Z^{\mathrm{no}}$ for every height 1 prime $\mathfrak{P} \subseteq \Lambda_{\mathbf{H}}$ not containing (p). The right hand side of (5.9.1) is therefore equal to

$$\sum_{\mathfrak{P}} \mu_{\mathfrak{s}}(\mathfrak{P}) \operatorname{lgth}_{\mathfrak{P}} Z^{\mathrm{no}} + \operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_{\Sigma}(W^{\mathrm{no}})^{\vee}[\Theta].$$

We now note $Z^{\mathrm{no}}[\Theta] = 0$: $\operatorname{Sel}^{\Sigma}(T^{\mathrm{no},*})$ is a torsion-free $\Lambda_{\mathbf{H}}$-module by Proposition 4.7, so by the Snake Lemma $Z^{\mathrm{no}}[\Theta]$ is an $\mathbf{H}$-torsion submodule of the free $\mathbf{H}$-module $\langle z(1) \rangle / \Theta \langle z(1) \rangle$. In combination with the bound

$$\operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_{\Sigma}(W^{\mathrm{no}})^{\vee}[\Theta] \geq \operatorname{lgth}_{\mathfrak{s}} \mathrm{H}^2(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^{\mathrm{no},*})[\Theta]$$

of Lemma 5.6, we thereby get

$$\begin{aligned} & \sum_{\mathfrak{P}} \mu_{\mathfrak{s}}(\mathfrak{P}) \operatorname{lgth}_{\mathfrak{P}} Z^{\mathrm{no}} + \operatorname{lgth}_{\mathfrak{s}} \operatorname{Sel}_{\Sigma}(W^{\mathrm{no}})^{\vee}[\Theta] \geq \\ & \qquad \sum_{\mathfrak{P}} \mu_{\mathfrak{s}}(\mathfrak{P}) \operatorname{lgth}_{\mathfrak{P}} Z^{\mathrm{no}} + \operatorname{lgth}_{\mathfrak{s}} Z^{\mathrm{no}}[\Theta] + \operatorname{lgth}_{\mathfrak{s}} \mathrm{H}^2(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^{\mathrm{no},*})[\Theta]. \end{aligned}$$

Another application of Lemma 5.5 and the bound of Lemma 5.7 then gives

$$\begin{aligned} & \sum_{\mathfrak{P}} \mu_{\mathfrak{s}}(\mathfrak{P}) \operatorname{lgth}_{\mathfrak{P}} Z^{\mathrm{no}} + \operatorname{lgth}_{\mathfrak{s}} Z^{\mathrm{no}}[\Theta] + \operatorname{lgth}_{\mathfrak{s}} \mathrm{H}^2(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^{\mathrm{no},*})[\Theta] \\ & \qquad \geq \operatorname{lgth}_{\mathfrak{s}} Z^{\mathrm{no}}/\Theta Z^{\mathrm{no}} + \operatorname{lgth}_{\mathfrak{s}} \mathrm{H}^2(\mathbf{Q}_{\Sigma}/\mathbf{Q}, T^{\mathrm{no},*})[\Theta] \\ & \qquad \geq \operatorname{lgth}_{\mathfrak{s}} Z, \end{aligned}$$

which completes the proof. □

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