

GAGA Talk (SArS Fall 2004)

Outline:

- I. Definitions/Theorems
- II. Proof in the projective case
- III. Proof in the proper case
- IV. Applications

Refs: Serre's original paper "Géométrie Algébrique et Géométrie Analytique"
SGA 1 exp. XII

I. Definitions/Theorems

Def 1: 1) An alg. \mathbb{C} -scheme X is a scheme locally of fin. type / \mathbb{C} (think of an alg. variety if this frightens you)

2) A \mathbb{C} -analytic space (X, \mathcal{O}_X) is a space locally ringed by \mathbb{C} -algebras (\mathbb{C} -LRS) that is locally isomorphic to a \mathbb{C} -LRS of the form $(U, i^*(\mathcal{O}_U/\mathcal{I}))$, where $C \subseteq U$ is closed, $U \subseteq \mathbb{C}^n$ is open, and $\mathcal{I} \subseteq \mathcal{O}_U$ is coherent. Maps of analytic spaces are just maps of \mathbb{C} -LRS. (This is just a generalization of \mathbb{C} -manifold which allows singularities, nilpotents, non-Hausdorffness, etc...)

Given an alg. \mathbb{C} -scheme, define a functor $\Phi_X: (\text{an spaces}) \rightarrow (\text{sets})$ by $\Phi_X(Y) := \text{Hom}_{\mathbb{C}\text{-LRS}}(Y, X)$.

Thm 1: Φ_X is representable, i.e. \exists an. space X^{an} & morphism $\phi: X^{\text{an}} \rightarrow X$ of \mathbb{C} -LRSs satisfying the universal property: \forall an. space Y & morphism of \mathbb{C} -LRSs $\psi: Y \rightarrow X$, $\exists!$ morphism $\chi: Y \rightarrow X^{\text{an}}$ making $\begin{array}{ccc} X^{\text{an}} & \xrightarrow{\phi} & X \\ \chi \uparrow & & \psi \\ Y & & \end{array}$ commute.

Moreover, ϕ induces a bijection between X^{an} and $X(\mathbb{C})$ and for any $x \in X^{\text{an}}$, the map $\phi_x: \mathcal{O}_{X, \phi(x)} \rightarrow \mathcal{O}_{X^{\text{an}}, x}$ of local rings induces an isomorphism on completions $\hat{\phi}_x: \hat{\mathcal{O}}_{X, \phi(x)} \xrightarrow{\cong} \hat{\mathcal{O}}_{X^{\text{an}}, x}$. Thus ϕ_x is faithfully flat.

Remark: The term "flat" was invented by Serre to describe this map.

This gives a functor $\Phi: (\text{alg. } \mathbb{C}\text{-sch.}) \rightarrow (\text{an.sp.}) : X \mapsto X^{\text{an}}$

This functor commutes with fiber products, i.e. $(X \times_Y Z)^{\text{an}} = X^{\text{an}} \times_{Y^{\text{an}}} Z^{\text{an}}$.

[This follows from the fact that the scheme $X \times_Z Y$ is also the fiber product in the category of LRSs.]

Now take $\phi: X^{\text{an}} \rightarrow X$. Then for any \mathcal{O}_X -module \mathcal{F} , we can consider the $\mathcal{O}_{X^{\text{an}}}$ -module $\phi^* \mathcal{F} =: \mathcal{F}^{\text{an}}$. A thm. of Oka states that $\mathcal{O}_{X^{\text{an}}}$ is coherent, so that \mathcal{F} coh. $\Rightarrow \mathcal{F}^{\text{an}}$ coh.

Thm 2: The functor $\mathcal{F} \mapsto \mathcal{F}^{\text{an}}$ is exact & faithful.

Thm 3: X has one of the following properties $\Leftrightarrow X^{\text{an}}$ does

- | | | |
|----------------------|---------------------|------------------|
| (i) non-empty | (v) normal | (ix) irreducible |
| (ii) discrete | (vi) reduced | |
| (iii) Cohen-Macaulay | (vii) n-dimensional | |
| (iv) regular | (viii) connected | |

Thm 4: $f: X \rightarrow Y$ (f.type) is one of the following $\Leftrightarrow f^{\text{an}}: X^{\text{an}} \rightarrow Y^{\text{an}}$ is

- | | | |
|-------------------------------------|------------------|-------------------------|
| (i) flat | (v) normal | (x) an isomorphism |
| (ii) unramified | (vi) reduced | (xi) a monomorphism |
| (iii) étale (i.e. locally biholom.) | (vii) injective | (xii) an open immersion |
| (iv) smooth (i.e. submersive) | (viii) separated | |

- | | |
|--------------------------|---------------|
| (xiii) dominant | (xvii) finite |
| (xiv) a closed immersion | |
| (xv) an immersion | |
| (xvi) proper | |

(coherent)
↓

We can now state Serre's famous GAGA thms. which compare the sheaves on X to the sheaves on X^{an} when X/\mathbb{C} is proper.

Thm 5: (Serre-Grothendieck) Let X/\mathbb{C} be proper. Then:

- (1) For every coherent \mathcal{O}_X -module \mathcal{F} , there is a functorial isomorphism $\xi: H^q(X, \mathcal{F}) \rightarrow H^q(X^{an}, \mathcal{F}^{an})$ commuting with the long exact sequence.
- (2) The functor $(\text{coh. } \mathcal{O}_X\text{-mods}) \rightarrow (\text{coh. } \mathcal{O}_{X^{an}}\text{-mods}): \mathcal{F} \mapsto \mathcal{F}^{an}$ is an equivalence of (abelian) categories.

Remk: More generally, if $f: X \rightarrow Y$ is a proper morphism of alg. \mathbb{C} -sch., then there is an isom. $H^q: (R^q f_* \mathcal{F})^{an} \rightarrow R^q f_*^{an}(\mathcal{F}^{an})$ (same proof).

Cor: (Chow's thm.) Every closed analytic subspace of $\mathbb{P}_{\mathbb{C}}^n$ is algebraic.

Pf: Note that \mathbb{C} -projective space actually is $(\mathbb{P}_{\mathbb{C}}^n)^{an}$ (they represent the same functors ...). For clarity, write $X = \mathbb{P}_{\mathbb{C}}^n$ (alg. side).

Suppose that $Z^{an} \subseteq X^{an}$ is a closed subspace. Then it corresponds to a coherent ideal sheaf $\mathcal{I}_{Z^{an}} \subseteq \mathcal{O}_{X^{an}}$. By (2) above, we get $\mathcal{I}_Z \subseteq \mathcal{O}_X$ coh. \mathcal{I} a corresp. $Z \subseteq X$ closed subvar. whose analytification is easily seen to be Z^{an} . □

Time permitting, we'll also prove:

Thm. 6: Let X be an alg. \mathbb{C} -scheme, $\phi: X^{an} \rightarrow X$ the canonical map.

Then for $x \in X^{an}$, there is a canonical isom. between $\pi_1^{ét}(X, \phi(x))$ and the profinite completion of $\pi_1(X, x)$.

Cor: The functor (proper \mathbb{C} -sch.) \rightarrow (\mathbb{C} -an. sp.) is fully faithful, i.e. $\text{Hom}_{\mathbb{C}}(X, Y) = \text{Hom}(X^{an}, Y^{an})$.

II. Proof in the projective case.

We first need to prove the existence of X^{an} , i.e., that $\mathbb{F}_X := \text{Hom}_{\text{LRS}}(-, X)$ is representable. We also want that $\phi_x: \hat{\mathcal{O}}_{X, \phi(x)} \xrightarrow{\cong} \hat{\mathcal{O}}_{X^{an}, x}$.

Pf: (of Thm. 1) Let $X = \text{alg. } \mathbb{C}\text{-sch.}$

First we show that the problem is local on X , i.e., X^{an} exists \Leftrightarrow for any open cover $\{U_i\}$, each U_i^{an} exists.

" \Rightarrow ": Say we have $\phi: X^{\text{an}} \rightarrow X$. We claim $U_i^{\text{an}} = \phi^{-1}(U_i)$. In fact,

a map from an an.sp. $\mathcal{X} \rightarrow U_i$ gives a comm. diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\phi} & X \\ & \searrow & \uparrow \\ & & U_i \end{array} \quad \phi^{-1}(U_i), \text{ thus it does so as a map of analytic spaces.}$$

" \Leftarrow ": By above, solutions on the U_i restrict to solns on the overlaps, so the universal property allows us to glue uniquely to get X^{an} .

Now we directly construct $(\mathbb{A}_{\mathbb{C}}^n)^{\text{an}}$. No surprise, it's \mathbb{C}^n :

Giving a map of LRSs $\mathcal{X} \rightarrow \mathbb{A}_{\mathbb{C}}^n$ is the same as choosing n global sections in $T(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ (cf. [H] Ch. II Ex.), i.e. n holomorphic fns.

on \mathcal{X} . This is the same as a map of an. spaces $\mathcal{X} \rightarrow \mathbb{C}^n$. $\therefore \mathbb{C}^n \cong (\mathbb{A}_{\mathbb{C}}^n)^{\text{an}}$.

Finally, we just need to show that if X^{an} exists, then so does Z^{an} for any closed subscheme $Z \subseteq X$. (Since every alg. \mathbb{C} -sch. has a covering by closed subschemes of $\mathbb{A}_{\mathbb{C}}^n$). This corresp. to a coh. ideal $\mathcal{I}_Z \subseteq \mathcal{O}_X$. The ideal

$\phi^{-1}\mathcal{I}_Z \cdot \mathcal{O}_{X^{\text{an}}}$ is also coherent, so defines $Z' \subseteq X^{\text{an}}$ closed subspace.

Easy to check $Z' = Z^{\text{an}}$.

The question of $\hat{\phi}_x: \hat{\mathcal{O}}_{X, \phi(x)} \xrightarrow{\cong} \hat{\mathcal{O}}_{X^{\text{an}}, x}$ is clearly local, so we may assume that $X \subseteq \mathbb{A}_{\mathbb{C}}^n$ (closed subvar.). Clearly $\hat{\phi}_x: \hat{\mathcal{O}}_{\mathbb{A}_{\mathbb{C}}^n, \phi(x)} \xrightarrow{\cong} \hat{\mathcal{O}}_{\mathbb{C}^n, x}$ (this is just saying that $\mathbb{C}[x_1, \dots, x_n]^{\wedge} \cong \mathbb{C}\{x_1, \dots, x_n\}^{\wedge}$). The general case follows from this, since for any ideal $I \subseteq \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n, \phi(x)}$, we get $\hat{\mathcal{O}}_{\mathbb{A}_{\mathbb{C}}^n, \phi(x)} / I \hat{\mathcal{O}}_{\mathbb{A}_{\mathbb{C}}^n, \phi(x)} \xrightarrow{\cong} \hat{\mathcal{O}}_{\mathbb{C}^n, x} / I \cdot \hat{\mathcal{O}}_{\mathbb{C}^n, x}$.

(The fact that $X^{\text{an}} \leftrightarrow X(\mathbb{C})$ is clear from the univ. property). \square

Pf. (of Thm. 2) Exactness of $\mathcal{F} \mapsto \mathcal{F}^{an}$ follows from the flatness of $\phi: X^{an} \rightarrow X$.

To check faithfulness, \therefore only need check $\mathcal{F}^{an} = 0 \Rightarrow \mathcal{F} = 0$. But to check that $\mathcal{F} = 0$, it suffices to check on stalks at closed points (X is Jacobson), so this follows from the fact that $\phi_x: \mathcal{O}_{X, \phi(x)} \rightarrow \mathcal{O}_{X^{an}, x}$ is faithfully flat. \square

Now we give a complete sketch of Serre's thm. (Thm. 5) in the case that X/\mathbb{C} is projective. We start with (1).

Firstly, we need to describe $\varepsilon: H^q(X, \mathcal{F}) \rightarrow H^q(X^{an}, \mathcal{F}^{an})$. These maps just come from the fact that $\mathcal{F} \mapsto \mathcal{F}^{an}$ is exact & that $H^q(X, -)$ is a universal δ -functor.

Note that $(\mathbb{P}_{\mathbb{C}}^n)^{an} = \mathbb{P}^n$ and that $\mathcal{O}(1)^{an} = \mathcal{O}(1)$ ($\mathcal{O}(1)$ is the univ. inv. sheaf on \mathbb{P}^n)

We claim it suffices to prove (1) in the case that $X = \mathbb{P}^n$:

In fact, if we have $i: X \hookrightarrow \mathbb{P}^n$, then $i^{an}: X^{an} \rightarrow \mathbb{P}^n$ is also a closed immersion (we showed this in our existence proof). Thus there are canonical functorial isomorphisms $H^q(X, \mathcal{F}) \cong H^q(\mathbb{P}_{\mathbb{C}}^n, i_* \mathcal{F})$ (note that $i_* \mathcal{F}$ is again coherent \because i proper) and $H^q(X^{an}, \mathcal{F}^{an}) \cong H^q(\mathbb{P}_{an}^n, i_*^{an}(\mathcal{F}^{an}))$. Thus just need $i_*^{an}(\mathcal{F}^{an}) = (i_* \mathcal{F})^{an}$ (canonically). (This is easy if you think about it right). Claim is proved by uniqueness of ε .

\therefore WLOG, assume $X = \mathbb{P}^n$.

First note that $\varepsilon: H^q(X, \mathcal{F}) \rightarrow H^q(X^{an}, \mathcal{F}^{an})$ gives an isomorphism when $\mathcal{F} = \mathcal{O}_X$:

$$\text{indeed } \varepsilon: H^0(X, \mathcal{O}_X) \xrightarrow{\cong} H^0(X^{an}, \mathcal{O}_{X^{an}}) \text{ and } H^q(X, \mathcal{O}_X) = 0 = H^q(X^{an}, \mathcal{O}_{X^{an}}) \quad q \geq 1$$

$\mathbb{C} \quad \quad \quad \mathbb{C}$

\uparrow
 (Thm of Dolbeault
 $\Rightarrow H^q(X^{an}, \mathcal{O}_{X^{an}}) \cong H^{q,0}(X)$)

Now we prove (1) when $\mathcal{F} \cong \mathcal{O}_X(k)$

Proceed by induction on n ($X = \mathbb{P}^n$). The case $n=0$ is clear.

We have an exact sequence $0 \rightarrow \mathcal{O}_X(-1) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_E \rightarrow 0$ ($\mathcal{O}_E \subseteq \mathcal{O}_X$ a hyperplane)

Get $\forall k \in \mathbb{Z}$ an e.s. $0 \rightarrow \mathcal{O}_X(k-1) \rightarrow \mathcal{O}_X(k) \rightarrow \mathcal{O}_E(k) \rightarrow 0$

\therefore have comm. diagram

$$\begin{array}{ccccccc}
 \cdots \rightarrow H^q(X, \mathcal{O}(k-1)) & \rightarrow & H^q(X, \mathcal{O}(k)) & \rightarrow & H^q(E, \mathcal{O}_E(k)) & \rightarrow & H^{q+1}(X, \mathcal{O}(k-1)) \rightarrow \cdots \\
 & & \downarrow & & \downarrow \cong & & \downarrow \\
 \cdots \rightarrow H^q(X^{an}, \mathcal{O}(k-1)^{an}) & \rightarrow & H^q(X^{an}, \mathcal{O}(k)^{an}) & \rightarrow & H^q(E^{an}, \mathcal{O}_{E^{an}}(k)^{an}) & \rightarrow & H^{q+1}(X^{an}, \mathcal{O}(k-1)^{an}) \rightarrow \cdots
 \end{array}$$

(inductive hypoth.)

\therefore by the 5-lemma (1) is true for $\mathcal{O}(k) \Leftrightarrow$ it is true for $\mathcal{O}(k-1)$

\therefore It is true $\forall k$ since it's true for $k=0$.

Finally, any coh. sheaf \mathcal{F} on X is the quotient of a sheaf $\mathcal{L} \cong \bigoplus_i \mathcal{O}(k_i)$

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{L} \rightarrow \mathcal{F} \rightarrow 0 \quad (\mathcal{H} \text{ coherent})$$

Get a comm. diagram

$$\begin{array}{ccccccccc}
 \cdots \rightarrow H^q(X, \mathcal{H}) & \rightarrow & H^q(X, \mathcal{L}) & \rightarrow & H^q(X, \mathcal{F}) & \rightarrow & H^{q+1}(X, \mathcal{H}) & \rightarrow & H^{q+1}(X, \mathcal{L}) \rightarrow \cdots \\
 \varepsilon_1 \downarrow & & \varepsilon_2 \downarrow & & \varepsilon_3 \downarrow & & \varepsilon_4 \downarrow & & \varepsilon_5 \downarrow \\
 \cdots \rightarrow H^q(X^{an}, \mathcal{H}^{an}) & \rightarrow & H^q(X^{an}, \mathcal{L}^{an}) & \rightarrow & H^q(X^{an}, \mathcal{F}^{an}) & \rightarrow & H^{q+1}(X^{an}, \mathcal{H}^{an}) & \rightarrow & H^{q+1}(X^{an}, \mathcal{L}^{an}) \rightarrow \cdots
 \end{array}$$

We know that ε_2 & ε_5 are isomorphisms.

Argue by descending induction on q : if $q > 0$, ε_4 is an isomorphism since

$$H^{q+1}(X, \mathcal{H}) = 0 = H^{q+1}(X^{an}, \mathcal{H}^{an}).$$

If $\varepsilon_2, \varepsilon_4, \varepsilon_5$ are isomorphisms, then the 5-lemma $\Rightarrow \varepsilon_3$ is surjective.

But then we've proved this for all coh. sheaves, so that ε_1 is also surjective.

Another application of the 5-lemma $\Rightarrow \varepsilon_3$ is injective \therefore an isom.

\therefore (1) is true for all coherent \mathcal{F} .

Now to prove (2): $\mathcal{F} \mapsto \mathcal{F}^{an}$ gives an equiv. of categories $(\text{coh. } \mathcal{O}_X\text{-mods.}) \cong (\text{coh. } \mathcal{O}_{X^{an}}\text{-mods.})$

$\mathcal{F} \mapsto \mathcal{F}^{an}$ is fully faithful: already know exact & faithful.

We can identify the map $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \rightarrow \text{Hom}_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an}, \mathcal{G}^{an})$

with $H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \rightarrow H^0(X^{an}, \mathcal{H}om_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an}, \mathcal{G}^{an}))$

The natural map $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^{an} \rightarrow \mathcal{H}om_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an}, \mathcal{G}^{an})$ is an isomorphism, since the map $\phi: X^{an} \rightarrow X$ is flat. Thus (1) gives the result, since $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is coh.

We now just need to prove that $\mathcal{F} \mapsto \mathcal{F}^{an}$ is essentially surjective (in the case X is projective).

By the same reasoning used in the pf. of (1), suff. to consider $X = \mathbb{P}^n$.

We proceed by induction on n . (The case $n=0$ is clear).

Claim 1: If $E \subset X$ is a hyperplane and \mathcal{A} is a coh. sheaf on E^{an} , then for all $q > 0$, and $k \gg 0$, $H^q(E^{an}, \mathcal{A}(k)) = 0$.

Pf: by ind. hyp. $\exists \mathcal{F}$ coh. on E s.t. $\mathcal{F}^{an} = \mathcal{A}$, so that $\mathcal{F}(k)^{an} = \mathcal{A}(k) \not\cong$ then pt. (1) gives the claim.

Claim 2: If \mathcal{M} is coh. on X^{an} , then $\exists k_0 = k_0(\mathcal{M})$ s.t. $\forall k \geq k_0$ $\mathcal{M}(k)$ is globally generated.

Pf: In fact if $\mathcal{M}(k)$ is gl. gen., then $\mathcal{M}(k)$ is too $\forall k \geq k_0$, since $\forall i \exists$ map $\mathcal{M}(k) \rightarrow \mathcal{M}(k)$ which is an isom on U_i . (mult. by t_i)

Also, by compactness of X^{an} , it suffices to prove that for any fixed $x \in X^{an}$, $\exists k_0$ s.t. $H^0(X^{an}, \mathcal{M}(k_0)) \rightarrow \mathcal{M}(k_0)_x$.

Choosing a hyperplane E through x gives an e.s.

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}_{X^{an}} \rightarrow \mathcal{O}_{E^{an}} \rightarrow 0$$

Tensor w/ \mathcal{M} :

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{M}(-1) \rightarrow \mathcal{M} \rightarrow i^* \mathcal{M} \rightarrow 0 \quad (i: E^{an} \hookrightarrow X^{an})$$

$\therefore \forall k$ get

$$0 \rightarrow \mathcal{K}(k) \rightarrow \mathcal{M}(k-1) \rightarrow \mathcal{M}(k) \rightarrow i^* \mathcal{M}(k) \rightarrow 0$$

Which breaks up as

$$0 \rightarrow \mathcal{K}(k) \rightarrow \mathcal{M}(k-1) \rightarrow \mathcal{F}_k \rightarrow 0, \quad 0 \rightarrow \mathcal{F}_k \rightarrow \mathcal{M}(k) \rightarrow i^* \mathcal{M}(k) \rightarrow 0$$

So we get from the l.e.s.

$$H^1(X^{an}, \mathcal{M}(k-1)) \rightarrow H^1(X^{an}, \mathcal{F}_k) \rightarrow H^2(X^{an}, \mathcal{K}(k))$$

$$\not\cong H^1(X^{an}, \mathcal{F}_k) \rightarrow H^1(X^{an}, \mathcal{M}(k)) \rightarrow H^1(X^{an}, i^* \mathcal{M}(k))$$

Both $\mathcal{K} \not\cong i^* \mathcal{M}$ are sheaves on E^{an} , so by claim (1), the higher cohom. of $\mathcal{K}(k) \not\cong i^* \mathcal{M}(k)$ vanishes $\forall k \gg 0$.

The dimension of these groups is finite by an analytic result of Serre-Cartan.

\therefore we see that the dim. of $H^1(X^{an}, \mathcal{M}(k))$ is a decreasing function of k for $k \gg 0$, hence constant for $k \gg 0$, and we get that the map

$H^1(X^{an}, \mathcal{F}_k) \rightarrow H^1(X^{an}, \mathcal{M}(k))$ is an isom. $\forall k \gg 0$, so the map

$H^0(X^{an}, \mathcal{M}(k)) \rightarrow H^0(X^{an}, i^* \mathcal{M}(k))$ is surjective $\forall k \gg 0$.

Then choose k_0 s.t. $i^*M(k_0)_X$ is generated by global sections
(ind. hypoth. $\Rightarrow i^*M$, considered as a sheaf on E , is algebraic ...)

We want to show that this k_0 works.

In fact, $i^*M(k_0)_X = M(k_0)_X \otimes_{\mathcal{O}_{X^{\text{an}}, X}} \mathcal{O}_{E^{\text{an}}, X} = M(k_0)_X / \mathcal{P}_E M(k_0)_X$ where

$\mathcal{P}_E = (\mathcal{I}_{E^{\text{an}}})_X$ satisfies $\mathcal{O}_{X^{\text{an}}, X} / \mathcal{P}_E = \mathcal{O}_{E^{\text{an}}, X}$. Let $\mathfrak{m}_X \in \mathcal{O}_{X^{\text{an}}, X}$ be the max. ideal.

Thus we have $H^0(X^{\text{an}}, M(k_0)) \rightarrow H^0(X^{\text{an}}, i^*M(k_0)) \rightarrow M(k_0)_X / \mathcal{P}_E M(k_0)_X \rightarrow M(k_0)_X / \mathfrak{m}_X M(k_0)_X$

Then NAK \Rightarrow the image of $H^0(X^{\text{an}}, M(k_0))$ generates $M(k_0)_X$. Claim proved.

Using these claims we can finish the pf. of (2):

By 2nd claim, any M coh. on X^{an} has that $M(k)$ is a qt. of $\mathcal{O}_{X^{\text{an}}}^d$ for some k, d . Thus \exists surjection $\mathcal{O}_{X^{\text{an}}}(-k)^d \rightarrow M$.

Applying the same reasoning to the kernel of this, we get an exact sequence

$$\mathcal{O}_{X^{\text{an}}}(-k')^{d'} \rightarrow \mathcal{O}_{X^{\text{an}}}(-k)^d \rightarrow M \rightarrow 0$$

The first two terms come from sheaves on $X \setminus \xi$ so does the map between them
Hence M is also algebraic. \square

Cor: If $\sigma \in \text{Aut } \mathbb{C}$, X and X^σ have the same Betti numbers.

III. Proof in the proper case

We deduce Thm. 5 in the case X/\mathbb{C} is proper from the projective case by using the following general

Lemma: (dévissage, EGA III, 3.1.2) $X = \text{Noeth. scheme}$. Let $K = (\text{coh. } \mathcal{O}_X\text{-mods.})$

Let $K' \subseteq \text{Obj}(K)$ be an exact subset (i.e. for any e.s. $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ if any two terms are in K' , so is the third) which contains, for every $x \in X$ a coh. \mathcal{O}_x -mod. \mathcal{F}_x such that $\mathcal{F}_x \neq 0$ and which is stable under taking direct factors. Then $K' = \text{Obj}(K)$.

By the dévissage lemma, if X/\mathbb{C} is proper, then to prove Thm. 5 (1) we need that the subset $K' \subseteq (\text{coh. } \mathcal{O}_X\text{-mods.})$ consisting of sheaves \mathcal{F} such that $\varepsilon: H^q(X, \mathcal{F}) \rightarrow H^q(X^{an}, \mathcal{F}^{an})$ is an isom. $\forall q$ satisfies the hypotheses of the above lemma.

- K' is exact: if $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ is exact, then consider the comm. diagram:

$$\begin{array}{cccccccc} \dots & \rightarrow & H^q(X, \mathcal{F}) & \rightarrow & H^q(X, \mathcal{G}) & \rightarrow & H^q(X, \mathcal{H}) & \rightarrow & H^{q+1}(X, \mathcal{F}) & \rightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \rightarrow & H^q(X^{an}, \mathcal{F}^{an}) & \rightarrow & H^q(X^{an}, \mathcal{G}^{an}) & \rightarrow & H^q(X^{an}, \mathcal{H}^{an}) & \rightarrow & H^{q+1}(X^{an}, \mathcal{F}^{an}) & \rightarrow & \dots \end{array}$$

By the 5-lemma, then, if any two of $\mathcal{F}, \mathcal{G}, \mathcal{H}$ is in K' , so is the third.

- K' is stable under direct factors: this is likewise clear from the comm. diagram:

$$\begin{array}{ccc} H^q(X, \mathcal{F} \oplus \mathcal{G}) & \xrightarrow{\cong} & H^q(X^{an}, \mathcal{F}^{an} \oplus \mathcal{G}^{an}) \\ \parallel & & \parallel \\ H^q(X, \mathcal{F}) \oplus H^q(X, \mathcal{G}) & \rightarrow & H^q(X^{an}, \mathcal{F}^{an}) \oplus H^q(X^{an}, \mathcal{G}^{an}) \end{array}$$

- The hard part is finding $\mathcal{F} \in K'$ st. $\mathcal{F}_x \neq 0$. Clearly we may assume X is irreducible with generic point x .

Now by Chow's lemma, $\exists g: X' \rightarrow X$ projective & birational st. X'/\mathbb{C} is projective

[Note that the notation $\mathcal{O}_x(n)$ is unambiguous ...]

By a thm. of Serre, $\exists n > 0$ st. $R^p g_* \mathcal{O}_x(n) = 0 \forall p > 0$ and $g^* g_* \mathcal{O}_x(n) \rightarrow \mathcal{O}_x(n)$

is surjective. We claim that the sheaf $\mathcal{F}_i := g_* \mathcal{O}_x(n)$ has $\mathcal{F}_i \neq 0$ and

$$\varepsilon: H^q(X, \mathcal{F}_i) \xrightarrow{\cong} H^q(X^{an}, \mathcal{F}_i^{an}) \forall q.$$

Now note $\mathcal{F}_i \neq 0$, since if $\xi \mapsto x$, then $(g^* \mathcal{F}_i)_\xi = \mathcal{F}_i \otimes_{\mathcal{O}_{X', \xi}} \mathcal{O}_{X', \xi}$ and $(g^* \mathcal{F}_i)_\xi \rightarrow \mathcal{O}_{X', \xi}$

is surjective by construction.

Recall the Leray spectral sequence $H^p(X, R^q g_* \mathcal{O}_x(n)) \Rightarrow H^{p+q}(X', \mathcal{O}_x(n))$.

This is degenerate by the fact that $R^p g_* \mathcal{O}_x(n) = 0 \forall p > 0$, so that

$$H^q(X, \mathcal{F}_i) \cong H^q(X', \mathcal{O}_x(n)).$$

We'll need to assume that we have proved the rnk. after Thm. 5 (i.e. that if $f: X \rightarrow Y$ is projective, then $(R^p f_* \mathcal{F})^{an} \cong R^p f_*^{an} \mathcal{F}^{an}$ functorially and that if Y is $\text{Spec } \mathbb{C}$, then this isom. is just ε of Thm. 5 (1)).

By this rnk., we have that $R^p g_*^{an} \mathcal{O}_{X'(n)}^{an} \cong (R^p g_* \mathcal{O}_{X'(n)})^{an} = 0$, so the same argument as above shows that $H^q(X, \mathcal{F}^{an}) \cong H^q(X'^{an}, \mathcal{O}_{X'(n)}^{an})$

Thus we can just look at the commutative diagram ^(edge maps in the Leray ss. are functorial)

$$\begin{array}{ccc} H^q(X, \mathcal{F}) & \cong & H^q(X', \mathcal{O}_{X'(n)}) \\ \varepsilon \downarrow & & \downarrow \varepsilon \\ H^q(X^{an}, \mathcal{F}^{an}) & \cong & H^q(X'^{an}, \mathcal{O}_{X'(n)}^{an}) \end{array}$$

The right-hand map is an isom. by what we've already proved. Thus the left-hand map is as well.

Now we prove Thm. 5 (2) (that $\mathcal{F} \mapsto \mathcal{F}^{an}$ is an equivalence of categories (coh. \mathcal{O}_X -mods.) \rightarrow (coh. $\mathcal{O}_{X^{an}}$ -mods.)).

We have shown unconditionally that $\mathcal{F} \mapsto \mathcal{F}^{an}$ is fully faithful.

Thus we need to deduce essential surjectivity from the projective case.

(this should really be located on the next page after "wlog.")

Again use Chow's lemma to find $f: X' \rightarrow X$ projective, surjective, birational with X'/\mathbb{C} projective. Let $U \subseteq X$ be a dense open such that $f|_U$ is an isomorphism.

We proceed by Noeth. induction on X : for a closed set $Y \subseteq X$, let $P(Y)$ be the statement: if \mathcal{G} is a coherent sheaf on X^{an} such that $\text{Supp } \mathcal{G} \subseteq Y^{an}$, then \exists coh. sheaf \mathcal{H} on X st. $\mathcal{H}^{an} \cong \mathcal{G}$.

We need to show that $P(Y')$ holds $\forall Y' \not\subseteq Y \Rightarrow P(Y)$ holds.

But note that if \mathcal{G} is any sheaf on X with $\text{Supp } \mathcal{G} \subseteq Y$, then by the analytic Nullstellensatz, there is a power of \mathcal{I}_Y^{an} that kills \mathcal{G} , so that we may view \mathcal{G} as a sheaf on $(Y, \mathcal{O}_X/\mathcal{I}_Y^{an})^{an}$.

Thus we may assume WLOG that $X=Y$.

Let \mathcal{F}^h be a coh. $\mathcal{O}_{X^{\text{an}}}$ -mod. \S consider the sequence (exact)

$$0 \rightarrow \mathcal{K}^h \rightarrow \mathcal{F}^h \xrightarrow{f_*^{\text{an}}} f_*^{\text{an}} \mathcal{F}^h \rightarrow \mathcal{L}^h \rightarrow 0$$

Since X' is projective, $f_*^{\text{an}} \mathcal{F}^h \cong \mathcal{G}$ for some \mathcal{G} coh. $\mathcal{O}_{X'}$ -mod

By hypothesis, $\mathcal{K}^h \cong \mathcal{K}^{\text{an}}$ and $\mathcal{L}^h \cong \mathcal{L}^{\text{an}}$ for some \mathcal{K} \S \mathcal{L} coh \mathcal{O}_X -mods, since

$$\mathcal{K}^h|_U = \mathcal{K}^{\text{an}}|_U = 0 \quad (\text{thm. 5(2)}) \quad (\text{def.})$$

Note that $(f_* \mathcal{G})^{\text{an}} \cong f_*^{\text{an}} \mathcal{G}^{\text{an}} \cong f_*^{\text{an}} f_*^{\text{an}} \mathcal{F}^h$, so that the map $f_*^{\text{an}} f_*^{\text{an}} \mathcal{F}^h \rightarrow \mathcal{L}^h$

(by the fact that the functor has already been proved to be fully faithful)

comes from an algebraic map $f_* \mathcal{G} \rightarrow \mathcal{L}$. If we set $\mathcal{J} = \ker(f_* \mathcal{G} \rightarrow \mathcal{L})$,

then we see (by exactness) that \mathcal{F}^h is an extension of \mathcal{J}^{an} by \mathcal{K}^{an} .

Thus it will suffice to prove that the map $\text{Ext}_{\mathcal{O}_X}^q(\mathcal{I}, \mathcal{K}) \rightarrow \text{Ext}_{\mathcal{O}_{X^{\text{an}}}}^q(\mathcal{I}^{\text{an}}, \mathcal{K}^{\text{an}})$

given concretely by mapping a q -extension \mathcal{H}_0 to $\mathcal{H}_0^{\text{an}}$ is an isomorphism \downarrow

(also cf. EGA 0_{III} §12.3)

Because the map of Ext sheaves $\text{Ext}_{\mathcal{O}_X}^q(\mathcal{J}, \mathcal{K})^{\text{an}} \rightarrow \text{Ext}_{\mathcal{O}_{X^{\text{an}}}}^q(\mathcal{J}^{\text{an}}, \mathcal{K}^{\text{an}})$ is an isom.

(since $\phi: X^{\text{an}} \rightarrow X$ is flat), the morphism of spectral sequences

$$\begin{array}{ccc} \text{HP}(X, \text{Ext}_{\mathcal{O}_X}^q(\mathcal{J}, \mathcal{K})) & \xrightarrow{\cong} & \text{Ext}_{\mathcal{O}_X}^{p+q}(\mathcal{J}, \mathcal{K}) \\ & & \downarrow \\ \text{HP}(X, \text{Ext}_{\mathcal{O}_{X^{\text{an}}}}^q(\mathcal{J}^{\text{an}}, \mathcal{K}^{\text{an}})) & \xrightarrow{\cong} & \text{Ext}_{\mathcal{O}_{X^{\text{an}}}}^{p+q}(\mathcal{J}^{\text{an}}, \mathcal{K}^{\text{an}}) \end{array}$$

is an isom on E_2 -terms and hence an isomorphism. D

Cor. If $X, Y/\mathbb{C}$ proper, then $\text{Hom}_{\mathbb{C}}(X, Y) \cong \text{Hom}(X^{\text{an}}, Y^{\text{an}})$

Pf. Look at the graph. D