

Math 1B03 Midterm 2

Monday 9 November 2009

1. Consider the lines ℓ_1 and ℓ_2 which have parametric equations

$$\ell_1 : \begin{cases} x(t) = -1 + t \\ y(t) = 1 + t \\ z(t) = t \end{cases} \quad \ell_2 : \begin{cases} x(t) = 2 \\ y(t) = 3 + 2t \\ z(t) = 4 \end{cases} .$$

(a) Find the equation of the plane r which is parallel to ℓ_1 and which contains ℓ_2 .

(b) Calculate the distance between ℓ_1 and ℓ_2 by calculating the distance from ℓ_1 to the plane r you found in part (a).

2. Find the equation of the plane containing the point $P = (1, 0, 1)$ which is parallel to the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} .$$

3. Write each of the following complex numbers in the form $a + bi$ for real numbers a and b .

(a) $((\cos \frac{\pi}{200}) + (\sin \frac{\pi}{200})i)^{100}$

(b) i^i

(c) $\frac{3 + 4i}{3 - 4i}$

4. Find all eigenvalues and eigenvectors for the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} .$$

5. Decide which of the following functions is a linear transformation. Prove your answers!

(a) The function $T : M_{2 \times 3} \rightarrow M_{2 \times 3}$ defined by $T(A) = \text{RREF}(A)$, i.e., T sends a 2×3 matrix to its reduced row echelon form.

(b) The function $\text{ev}_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} : \text{Hom}(\mathbf{R}^2, \mathbf{R}) \rightarrow \mathbf{R}$ defined by $\text{ev}_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}(T) = T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$, i.e., the input into $\text{ev}_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$ is a linear transformation $T : \mathbf{R}^2 \rightarrow \mathbf{R}$ and the output is the value of T on the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. (Recall that $\text{Hom}(\mathbf{R}^2, \mathbf{R})$ is the vector space of linear transformations from \mathbf{R}^2 to \mathbf{R} .)

(c) The function $R : P_2 \rightarrow P_2$ defined by $R(a + bx + cx^2) = x + xb + xb^2$.

(d) The function $R : P_2 \rightarrow P_2$ defined by $R(a + bx + cx^2) = b + bx + bx^2$.

6. Decide which of the following sets is a subspace of P_3 . Prove your answers!

(a) The set of polynomials $f(t)$ in P_3 such that $f(2) \geq 0$.

(b) The set of polynomials $f(t)$ in P_3 such that $f(t) = f(2 - t)$.

(c) The set of polynomials $f(t)$ in P_3 such that $f(t) = 2 - f(t)$.

7. Find the standard matrix associated to each of the following linear transformations. (Recall that the vector space P_n of polynomials of degree $\leq n$ has standard basis $\{1, t, t^2, \dots, t^n\}$.)

(a) $R : P_3 \rightarrow \mathbf{R}$ defined by $R(f(t)) = f(2)$

(b) $S^{-1} : P_1 \rightarrow P_1$, the inverse of the transformation S defined by $S(a + bt) = a + (a + b)t$

(c) $T : \mathbf{R}^3 \rightarrow \mathbf{R}$ defined by $T(\vec{x}) = \vec{u} \cdot (\vec{x} \times \vec{v})$ for $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

8. Decide whether each of the following transformations is injective (one-to-one), surjective (onto), and/or bijective. Show your work.

(a) $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y - 2z \\ 3y + 3z \\ 6y + 6z \end{pmatrix}$$

(b) $T : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$$

(c) $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by $T(\vec{x}) = 4\vec{x}$

9. In each part below, you are given a vector \vec{v} and a finite set of vectors S . Decide whether \vec{v} is in $\text{Span } S$. If it is, express \vec{v} as a linear combination of vectors in S .

(a) $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \right\}$

(b) $\vec{v} = \begin{pmatrix} 6 \\ -4 \\ -6 \\ 8 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 2 \\ 0 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(c) $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$