Math 1B03 Midterm 2

Solutions

1.(a) To find the equation of r, we make use of the facts:

- $\rightarrow$ r is parallel to ℓ_1 and ℓ_2, so in particular r is parallel to the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$
- $\rightarrow$ r contains every point on ℓ_2, and in particular the point $P = (2,3,4)$.

Use the first fact to find a normal vector $\vec{n} = \vec{v}_1 \times \vec{v}_2$ for r:

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix} = (-2)\vec{i} - (0)\vec{j} + (2)\vec{k} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$$

Thus r has equation $\quad (-2)(x-2) + (0)(y-3) + (2)(z-4) = 0$

i.e., $\quad r: \quad \underline{\underline{-2x + 2z - 4 = 0}}$.

(b) We calculate the distance from ℓ_1 to r:

choose a point on ℓ_1: choose $(-1,1,0)$

choose a point on r: choose $(2,3,4)$

The vector from $(-1,1,0)$ to $(2,3,4)$ is $\vec{u} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$.

From (a) we know that $\vec{n} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ is a normal vector for r.

The distance from ℓ_1 to r (and hence the distance between ℓ_1 and ℓ_2) is

$$\|\text{proj}_{\vec{n}}\vec{u}\| = \frac{|\vec{u}\cdot\vec{n}|}{\|\vec{n}\|} = \frac{|-6+8|}{\sqrt{(-2)^2+2^2}} = \frac{2}{\sqrt{8}} = \underline{\underline{\frac{1}{\sqrt{2}}}}$$

2. $\vec{v}_1 \times \vec{v}_2$ gives a normal vector for this plane:

$$\vec{v}_1 \times \vec{v}_2 = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = -4\vec{i} + 8\vec{j} - 4\vec{k}$$

The plane containing $(1,0,1)$ with normal vector $\begin{pmatrix} -4 \\ 8 \\ -4 \end{pmatrix}$ has equation:

$$-4(x-1) + 8(y-0) - 4(z-1) = 0, \text{ i.e., } \underline{\underline{-4x + 8y - 4z + 8 = 0}}$$

3. (a) $\left(\cos(\pi/200) + \sin(\pi/200)\, i\right)^{100} = \left(e^{(\pi/200)\, i}\right)^{100}$

$$= e^{100 \cdot (\pi/200)\, i} = e^{\pi/2\, i} = \cos\frac{\pi}{2} + \sin\frac{\pi}{2}\, i = \underline{\underline{0 + 1 \cdot i = i}}$$

(b) $i^i = \left(e^{\pi/2\, i}\right)^i = e^{(\pi/2\, i)\, i} = \underline{\underline{e^{-\pi/2}}}$

(c) $\dfrac{3+4i}{3-4i} = \dfrac{(3+4i)(3+4i)}{(3-4i)(3+4i)} = \dfrac{(9-16)+(12+12)\, i}{3^2+4^2} = -\dfrac{7}{25} + \dfrac{24}{25}\, i$

4.

$$\det\left(\lambda I_3 - A\right) = \det\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda-1 & 1 \\ 0 & -1 & \lambda-1 \end{pmatrix} = \lambda \det\begin{pmatrix} \lambda-1 & 1 \\ -1 & \lambda-1 \end{pmatrix} =$$

$$= \lambda\left((\lambda-1)(\lambda-1)+1\right) = \lambda\left(\lambda^2 - 2\lambda + 2\right)$$

The roots of this are $\lambda = 0$ and $\lambda = \frac{1}{2}\left(2 \pm \sqrt{4-8}\right) = 1 \pm i$

So the eigenvalues of A are: $\lambda = 0$, $\lambda = 1+i$, and $\lambda = 1-i$.

Eigenvectors for $\lambda = 0$: Solve $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1 \mapsto -r_1} \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\xrightarrow{r_2 \mapsto r_2 + r_1} \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_2 \mapsto \frac{1}{2} r_2} \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1 \mapsto r_1 - r_2} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Solutions: $\begin{array}{l} x_1 = t \\ x_2 = 0 \\ x_3 = 0 \end{array}$, i.e., $\vec{x} = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ are the eigenvectors with eigenvalue 0.

Eigenvectors for $\lambda = 1+i$:

Solve $\left(\begin{array}{ccc|c} 1+i & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & -1 & i & 0 \end{array}\right) \xrightarrow{r_1 \mapsto \frac{1}{1+i} r_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & -1 & i & 0 \end{array}\right) \xrightarrow{r_2 \mapsto -i r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -1 & i & 0 \end{array}\right)$

$\xrightarrow{r_3 \mapsto r_3 + r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ Solutions: $\begin{array}{l} x_1 = 0 \\ x_2 = t\, i \\ x_3 = t \end{array}$, i.e., $\vec{x} = t\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$ are the

eigenvectors with eigenvalue $1+i$

By taking complex conjugates, we see that $\vec{x} = t\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$ are the

eigenvectors with eigenvalue $1-i$

5. (a) not a linear transformation:

example:

$$\left.\begin{aligned} &T\left(\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}\right)=T\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\\ \text{but}\quad &T\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}+T\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&1&0\\0&0&0\end{pmatrix}=\begin{pmatrix}1&1&0\\0&0&0\end{pmatrix}\end{aligned}\right\}\text{ not equal}$$

(b) this is a linear transformation:

proof: addition:

suppose S and T are any two linear transformations. Then:

$$\left.\begin{aligned} &ev_{\binom{2}{1}}(S+T)=(S+T)\binom{2}{1}\\ \text{and}\quad &ev_{\binom{2}{1}}(S)+ev_{\binom{2}{1}}(T)=S\binom{2}{1}+T\binom{2}{1}\end{aligned}\right\}\text{ these are equal}$$

Scalar multiplication: Suppose a is any scalar and $T:\mathbb{R}^2\to\mathbb{R}^1$ a transformation

$$\left.\begin{aligned} &ev_{\binom{2}{1}}(aT)=(aT)\binom{2}{1}\\ \text{and}\quad &a\,ev_{\binom{2}{1}}(T)=a\left(T\binom{2}{1}\right)\end{aligned}\right\}\text{ these are equal}$$

(c) not a linear transformation:

example:

$$\left.\begin{aligned} &R((1+x)+(x+x^2))=R(1+2x+x^2)=x+2x+4x=7x\\ \text{but}\quad &R(1+x)+R(x+x^2)=(x+x+x)+(x+x+x)=6x.\end{aligned}\right\}\text{ not equal.}$$

(d) this is a linear transformation:

Proof: addition:

Suppose $a+bx+cx^2$ and $a'+b'x+c'x^2$ are any two polynomials in P_2.

Then:

$$R\left((a+bx+cx^2)+(a'+b'x+c'x^2)\right)=R\left((a+a')+(b+b')x+(c+c')x^2\right)$$

$$\left.\begin{aligned} &=(b+b')+(b+b')x+(b+b')x^2\\ \text{and}\quad &R(a+bx+cx^2)+R(a'+b'x+c'x^2)=b+bx+bx^2+b'+b'x+b'x^2\end{aligned}\right\}\text{ equal.}$$

Scalar multiplication: if k is any scalar and $a+bx+cx^2$ any element of P_2:

$$\left.\begin{aligned} &R(k(a+bx+cx^2))=R(ka+kbx+kcx^2)=kb+kbx+kbx^2\\ \text{and}\quad &k\,R(a+bx+cx^2)=k(b+bx+bx^2)\end{aligned}\right\}\text{ equal.}$$

6. (a) <u>not</u> a subspace:

example:

Consider the scalar $a = -3$ and the polynomial $f(t) = t$

Then $f(t)$ satisfies $f(2) = 2 \geq 0$

but $-3f(t)$ has $-3f(2) = (-3)(2) = -6$, which is not ≥ 0

(b) this <u>is</u> a subspace:

Proof: addition:

Say $f(t)$ satisfies $f(t) = f(2-t)$ and $g(t) = g(2-t)$

then $(f+g)(t) = f(t) + g(t) = f(2-t) + g(2-t) = (f+g)(2-t)$

scalar multiplication:

If a is a scalar and $f(t)$ satisfies $f(t) = f(2-t)$,

then $af(2-t) = af(t)$

(c) <u>not</u> a subspace:

example:

Consider the polynomial $f(t) = 1$ and $g(t) = 1$; then $f(t) = 2 - f(t)$ and $g(t) = 2 - g(t)$, but $f(t) + g(t) = 1 + 1 = 2$ does not satisfy $2 = 2 - 2$.

7. (a) $R(1) = 1$, $R(t) = 2$, $R(t^2) = 4$, $R(t^3) = 8$

So R has matrix $\begin{pmatrix} 1 & 2 & 4 & 8 \end{pmatrix}$

(b) $S(1) = 1 + t$ and $S(t) = t$, so S has matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and

so S^{-1} has matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

(c) $$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \left(\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}1\\0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \left(\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 0\\ 1 & 0 & 1\end{pmatrix}\right) = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}0\\-1\\0\end{pmatrix} = -1$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \left(\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \left(\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}\right) = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix} = 0$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \left(\begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\1\\1\end{pmatrix}\right) = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \left(\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 0 & 1\\ 1 & 0 & 1\end{pmatrix}\right) = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = 1$$

So T has matrix $\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$.

8. (a) T has matrix $\begin{pmatrix} 1 & 2 & -2 \\ 0 & 3 & 3 \\ 0 & 6 & 6 \end{pmatrix}$. The RREF of this matrix

is $\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This has a zero row, so T is not surjective.
Also, there is a free variable column, so T is not injective
(T is also not bijective).

(b) T has matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$, which has RREF $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

This has a zero row, but no free variable columns

Thus T is injective but neither surjective nor bijective

(c) T has matrix $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, which has RREF $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

This matrix has no zero rows and no free variable columns,

so T is injective, surjective, and bijective.

9. (a) Solve $\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 2 & 1 \end{array}\right) \xrightarrow{r_3 \mapsto r_3 - r_1} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & 0 & 0 \end{array}\right) \xrightarrow{r_3 \mapsto r_3 + 2r_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 & 4 \end{array}\right)$

$\xrightarrow{r_1 \mapsto r_1 - 2r_2} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 & 4 \end{array}\right) \xrightarrow{r_3 \mapsto \frac{1}{3} r_3} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4/3 \end{array}\right) \xrightarrow{r_2 \mapsto r_2 - r_3} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -3 \\ 0 & 1 & 0 & 0 & 2/3 \\ 0 & 0 & 1 & 0 & 4/3 \end{array}\right)$

$\xrightarrow{r_1 \mapsto r_1 + 2r_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1/3 \\ 0 & 1 & 0 & 0 & 2/3 \\ 0 & 0 & 1 & 0 & 4/3 \end{array}\right)$ solutions: $x_1 = -1/3 - 2t$, $x_2 = 2/3$, $x_3 = 4/3$, $x_4 = t$ (t any real number)

Thus, since t can be any real number, we can choose $t = 0$ and we get

$$\vec{v} = -\frac{1}{3}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \frac{4}{3}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix},$$ so $\vec{v}$ is in Span S.

(b) Solve $\left(\begin{array}{cc|c} 2 & 0 & 6 \\ 0 & 1 & -4 \\ -2 & 0 & -6 \\ 4 & 1 & 8 \end{array}\right) \xrightarrow{r_1 \mapsto \frac{1}{2} r_1} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \\ -2 & 0 & -6 \\ 4 & 1 & 8 \end{array}\right) \xrightarrow{r_3 \mapsto r_3 + 2r_1} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 4 & 1 & 8 \end{array}\right) \xrightarrow{r_4 \mapsto r_4 - 4r_1} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 1 & -4 \end{array}\right)$

$\xrightarrow{r_4 \mapsto r_4 - r_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ Solution: $x_1 = 3$, $x_2 = -4$

Thus $\vec{v} = 3\begin{pmatrix} 2 \\ 0 \\ -2 \\ 4 \end{pmatrix} - 4\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ is in span S.

(c) Solve $\left(\begin{array}{ccc|c} 1 & 4 & 0 & 1 \\ -1 & -4 & 1 & 1 \\ 2 & 8 & 0 & 5 \end{array}\right) \xrightarrow{r_3 \mapsto r_3 - 2r_1} \left(\begin{array}{ccc|c} 1 & 4 & 0 & 1 \\ -1 & -4 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right)$ There are no solutions.

Thus $\vec{v}$ does not belong to Span S.