Math 1B03 Midterm 1

Solutions

1. (a) RREF
(b) not RREF
(c) RREF
(d) RREF
(e) not RREF
(f) not RREF

2. Multiply both sides by the inverse of $\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}, \text{ so } A = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 6 \\ 7 & 14 & 21 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

3. Expand on the indicated row or column:

$$\det\begin{pmatrix} 0 & 2 & 0 & 5 & 0 \\ 2 & 5 & 1 & 7 & -6 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 4 & 0 & -1 & 2 \\ 1 & 0 & -1 & -1 & 9 \end{pmatrix} = -2\det\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 5 & 1 & -6 \\ 0 & 4 & 0 & 2 \\ 1 & 0 & -1 & 9 \end{pmatrix} = 4\det\begin{pmatrix} 2 & 1 & -6 \\ 0 & 0 & 2 \\ 1 & -1 & 9 \end{pmatrix}$$

$$= -8\det\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = 24.$$

4. (a) C is diagonal, so $\det C = 3\cdot(-1)\cdot 2 = -6$ (product of diagonal entries)

(b) Let's compute the cofactors:

$C_{11} = -2$ $\quad C_{12} = -(0)$ $\quad C_{13} = 0$

$C_{21} = -(4)$ $\quad C_{22} = 6$ $\quad C_{32} = -(0)$

$C_{31} = 5$ $\quad C_{32} = -(6)$ $\quad C_{33} = -3$

Thus

$$\text{adj } C \quad \begin{pmatrix} -2 & -4 & 5 \\ 0 & 6 & -6 \\ 0 & 0 & -3 \end{pmatrix}$$ (transpose of cofactor matrix)

(c) $$C^{-1} = \frac{1}{\det C}\text{adj } C = -\frac{1}{6}\begin{pmatrix} -2 & -4 & 5 \\ 0 & 6 & -6 \\ 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 & -5/6 \\ 0 & -1 & 1 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

5. $$\begin{pmatrix} 3 & -3 & 1 \\ -5 & 2 & 0 \\ 1 & 2 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \\ 8 \end{pmatrix}$$ This is not a multiple of $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, so $\vec{x}$ is <u>not</u> an eigenvector

$$\begin{pmatrix} 3 & -3 & 1 \\ -5 & 2 & 0 \\ 1 & 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 9 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix},$$ so $\vec{y}$ is an eigenvector with eigenvalue -3.

6. (a) $\left(\begin{array}{ccc|ccc} 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|ccc} 2 & 2 & 4 & 0 & 0 & 1 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \end{array}\right) \xrightarrow{r_1 \mapsto \frac{1}{2} r_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1/2 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \end{array}\right)$

$\xrightarrow{r_2 \mapsto \frac{1}{2} r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1/2 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \end{array}\right) \xrightarrow{r_1 \mapsto r_1 - r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \end{array}\right)$

$\xrightarrow{r_3 \mapsto r_3 - 2r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & -2 & 1 & -1 & 0 \end{array}\right) \xrightarrow{r_3 \mapsto -\frac{1}{2} r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 0 \end{array}\right)$

$\xrightarrow{r_2 \mapsto r_2 - 2r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 0 \end{array}\right).$

(b) From (a), we see that D is invertible and $D^{-1} = \begin{pmatrix} 0 & -1/2 & 1/2 \\ 1 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix}$

7. (a) FALSE:

set $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $EF = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ but $E \neq 0$ and $F \neq 0$.

(b) FALSE:

set $G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $\det G = 0$ and $\vec{b} \neq \vec{0}$, but the equation $G\vec{x} = \vec{b}$ has the solution $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(c) FALSE:

Consider the linear system $\begin{array}{l} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 2 \end{array}$.

The augmented matrix for this system is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right)$, which has as its RREF the matrix $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$, so this system has no solutions.

(d) FALSE:

set $H = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $HK = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is singular, but K is not singular.

(e) FALSE:

Set $L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then both L and M are in RREF and both have rank 1, but $L \neq M$.

8. Multiply both sides by N^{-1} to get

$$\vec{x} = \begin{pmatrix} -2 & -1 & 5 \\ 1 & 0 & -1 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

9. The augmented matrix for this system is $\left(\begin{array}{cc|c} -1 & 2 & 1 \\ 1 & -h & k \end{array}\right)$.

Let's do some row ops. to this: $\left(\begin{array}{cc|c} -1 & 2 & 1 \\ 1 & -h & k \end{array}\right) \xrightarrow{r_1 \mapsto -r_1} \left(\begin{array}{cc|c} 1 & -2 & -1 \\ 1 & -h & k \end{array}\right)$

$\xrightarrow{r_2 \mapsto r_2 - r_1} \left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 2-h & 1+k \end{array}\right)$.

(a) In order for this system to be inconsistent, we must have $2-h=0$ and $1+k \neq 0$, i.e., $h=2$ and $k \neq -1$

(b) The only condition needed to guarantee that there will be exactly 1 solution is $2-h \neq 0$, i.e., $h \neq 2$ (k can be anything)

(c) The only way there could be infinitely many solutions is if $2-h=0$ and $1+k=0$, i.e., $h=2$ and $k=-1$