p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

$\begin{cases} \text{converges if} & p > 1 \\ \text{diverges if} & p \leq 1 \end{cases}$

geometric series $\sum_{n=1}^{\infty} ar^{n-1}$

$\begin{cases} \text{converges if} & |r| < 1 \\ \text{diverges if} & |r| \geq 1 \end{cases}$

Recall: If $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n$ converges iff the sequence $S_n = \sum_{i=1}^{n} a_i$ of partial sums converges, iff the sequence S_n is bounded above.

$a_n \geq 0 \Rightarrow \{S_n\}$ is monotone increasing, i.e. $S_1 \leq S_2 \leq S_3 \leq \ldots$

$S_n \leq S_{n+1}$

Comparison Test.
(CT)

Consider series $\sum a_n$ and $\sum b_n$ with

$$0 \leq a_n \leq b_n \text{ for all } n.$$

$\sum b_n$ converges $\Rightarrow \sum a_n$ converges

$\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges

NOTE: Actually only need $0 \leq a_n \leq b_n$ for $n \geq \bar{N}$ for some $\bar{N}$ since a finite # of terms does NOT change convergence OR DIVERGENCE.

Example: $S = \sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

$$0 \leq \frac{1}{n} \leq \frac{\ln(n)}{n}, \quad \text{if } n \geq 3$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

January 24, 2020 11:42 AM

$\therefore$ S diverges by CT.

Example. $S = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+n+1}}$

$$0 \le \frac{1}{\sqrt{n^5+n+1}} \le \frac{1}{\sqrt{n^5}} = \frac{1}{n^{5/2}}$$

Since $\sum \frac{1}{n^{5/2}}$ p-series with $p = 5/2 > 1$ converges

$\therefore$ S converges by CT.

Example. $S = \sum_{n=1}^{\infty} \frac{1+\cos(n)}{n^2}$

$$0 \le 1 + \cos(n) \le 2$$

since $-1 \le \cos(n) \le 1$.

$$0 \le \frac{1+\cos(n)}{n^2} \le \frac{2}{n^2}$$

$\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

January 24, 2020 11:51 AM

(p-series with $p = 2 > 1$.) $\frac{1}{n^2}$

$\therefore$ S CONVERGES.

Limit Comparison TEST.
(LCT)

$\sum a_n$ and $\sum b_n$
with POSITIVE terms
i.e. $a_n > 0$, $b_n > 0$

If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ is a finite <u>positive</u> number.

then either <u>BOTH</u> series converge
OR
<u>BOTH</u> series diverge.

Why?

January 24, 2020 11:56 AM

$\because \lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$

$\therefore$ for LARGE n, say $n \geq \bar{N}$

$$\left| \frac{a_n}{b_n} - c \right| < 1$$

$$-1 < \frac{a_n}{b_n} - c < 1 \Rightarrow \quad a_n - cb_n < b_n$$

$$\Rightarrow \quad a_n < (1+c)\, b_n, \quad n \geq \bar{N}$$

$\therefore$ If $\sum b_n$ converges

$\Rightarrow \sum (1+c) b_n$ converges

$\Rightarrow \sum a_n$ converges

by CT.

If $\sum a_n$ diverges

$\Rightarrow \sum (1+c) b_n$ diverges

$\Rightarrow \sum b_n$ diverges!

by CT.

Example (LCT)

January 24, 2020 12:01 PM

$$S = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

Know

g.s. $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges.

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}}$$

$$\lim_{n\to\infty} \frac{2^n}{2^n-1}$$

$$\lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} = \underset{c}{\underset{\shortparallel}{1}} > 0$$

$\therefore$ S converges by LCT.

Example. $\sum_{n=1}^{\infty} \frac{(n^2+1)}{\sqrt{n^6+n}}$

for large n

$$n^2+1 \sim n^2$$
$$n^6+n \sim n^6$$

Suggests comparing with

$$\frac{n^2}{\sqrt{n^6}} = \frac{n^2}{n^3} = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{(n^2+1)/\sqrt{n^6+n}}{1/n}$$

$$= \lim_{n\to\infty} \frac{n^3+n}{\sqrt{n^6+n}} \quad \begin{matrix} /n^3 \\ /n^3 \end{matrix}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{\sqrt{1+\frac{1}{n^5}}} = 1 > 0.$$

Since $\sum \frac{1}{n}$ diverges, S diverges by LCT.

Example.

January 24, 2020 12:13 PM

LCT does NOT work, but CT worked.

$$S = \sum_{n=1}^{\infty} \frac{1+\cos(n)}{n^2}$$ (Showed it converges by CT).

But.

$$\lim_{n\to\infty} \frac{\left(\frac{1+\cos(n)}{n^2}\right)}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} 1+\cos(n)$$

this limit does NOT exist.

$\therefore$ LCT fails. (inconclusive).