§11.2 Series, cont'd.

$\sum_{i=1}^{\infty} a_i$ converges if the sequence of partial sums $\{S_n\}$

$$S_n = \sum_{i=1}^{n} a_i,$$ CONVERGES

and DIVERGES otherwise.

Geometric Series

$a \neq 0$, $r \in \mathbb{R}$. $(r^0 = 1)$

$$S_n = \sum_{i=1}^{n} a r^{i-1} = \frac{a(1-r^n)}{1-r}$$

$$\sum_{i=1}^{\infty} a r^{i-1} = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$\text{CONVERGENT} = \begin{cases} a/1-r & \text{if } |r|<1 \\ \text{DIVERGENT} & \text{if } |r| \geq 1. \end{cases}$$

Example:
$$\sum_{i=1}^{\infty} \frac{1}{4^i} = \sum_{i=1}^{\infty} \frac{1}{4}\left(\frac{1}{4}\right)^{i-1}$$

$a = \frac{1}{4}$ $r = \frac{1}{4} < 1$

$$= \frac{1}{4}\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{1}{4}\,\frac{4}{3} = \frac{1}{3}.$$

CONVERGENT.

EXAMPLE
$$\sum_{n=1}^{\infty} 2^n (3)^{1-n} = \sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^n$$
$$= \sum_{n=1}^{\infty} \cancel{3}\left(\frac{2}{\cancel{3}}\right)\left(\frac{2}{3}\right)^{n-1}$$

$a = 2$, $r = 2/3 < 1$.

$$= 2\left(\frac{1}{1-2/3}\right) = 6$$

CONVERGENT

Example: Repeating decimal.

$4.2\overline{38} = 4.23838383838\ldots$

Write as a ratio of integers

i.e. fraction

$$4.2\overline{38} = 4.2 + \frac{38}{10^3} + \frac{38}{10^5} + \frac{38}{10^7} + \cdots$$

$$= \frac{42}{10} + \frac{38}{10^3}\left(1 + \frac{1}{10^2} + \left(\frac{1}{10^2}\right)^2 + \cdots\right.$$

$$= \frac{42}{10} + \sum_{n=1}^{\infty} \frac{38}{10^3}\left(\frac{1}{10^2}\right)^{n-1}$$

$$a = {}^{38}/_{10^3} \qquad r = \frac{1}{10^2} < 1.$$

$$= \frac{42}{10} + \frac{38}{1000}\left(\frac{1}{1 - \frac{1}{100}}\right)$$

$$= \frac{42}{10} + \frac{38}{990} = \frac{2098}{495}.$$

Telescoping Sum.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

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$$S_n = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$ (lots of cancelation)

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

CONVERGENT.

TEST for DIVERGENCE.

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is DIVERGENT.

Why? Assume $\sum_{n=1}^{\infty} a_n$ CONVERGENT.

$\lim_{n\to\infty} S_n = L$, $\lim_{n\to\infty} S_{n-1} = L$

$$S_n - S_{n-1} = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i$$

$$= (a_1 + a_2 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_{n-1})$$

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$$S_n - S_{n-1} = a_n$$

$$\lim_{n\to\infty} (S_n - S_{n-1}) = L - L = 0.$$

$$\therefore \lim_{n\to\infty} a_n = 0.$$

$\therefore$ If $\lim_{n\to\infty} a_n \neq 0$, the series DIVERGES.

Example.

$$\sum_{n=1}^{\infty} \frac{n}{n+1}, \qquad a_n = \frac{n}{n+1}$$

$$= \frac{1}{1+\frac{1}{n}} \to 1 \text{ as } n \to \infty$$

Since $a_n \to 1 \neq 0$.

the series DIVERGES

Example. $\sum_{i=1}^{\infty} (-1)^n$

$$a_n = (-1)^n$$

$\lim_{n\to\infty} a_n \neq 0 \therefore$ Series diverges.

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BEWARE: If $\lim_{n\to\infty} a_n = 0$, the test is INCONCLUSIVE. It does NOT mean the series converges.

Example. $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ DIVERGES.

harmonic series.

even though $a_n = \frac{1}{n}$.

$\lim_{n\to\infty} a_n = 0$.

$$\sum_{i=1}^{n} \frac{1}{n} = \underbrace{\underbrace{1+\frac{1}{2}}_{S_2} + \underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}}}_{S_4}$$

$$+ \underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{> \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = 4\left(\frac{1}{8}\right)}$$

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$S_2 = 1 + \frac{1}{2}$

$\underset{\llcorner 2^2}{S_4} > 1 + \frac{1}{2} + \frac{1}{2} \qquad 1 + 2\left(\frac{1}{2}\right)$

$\underset{\llcorner 2^3}{S_8} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right)$

$S_{2^n} > 1 + n\left(\frac{1}{2}\right)$

$\lim_{n \to \infty} S_n = \infty$.

$\therefore \sum_{i=1}^{\infty} \frac{1}{n}$ DIVERGES.

Rules for Series.

If $\sum a_n$ and $\sum b_n$ are CONVERGENT, then.

So are $\sum (a_n + b_n)$

$\sum (a_n - b_n)$

$\sum c a_n \quad (c \in \mathbb{R}.)$

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(i) $\sum (a_n + b_n) = \sum a_n + \sum b_n$

ii $\sum (a_n - b_n) = \sum a_n - \sum b_n$

iii $\sum c\, a_n = c \sum a_n$.

<u>NOTE</u>: A finite # of terms does not affect the convergence or divergence of a series

If $\sum_{n=4}^{\infty} a_n$ CONVERGES, then $\sum_{n=1}^{\infty} a_n$ CONVERGES

$a_1 + a_2 + a_3 + a_4$ is finite.