

Recall: $\{a_n\}$ is an infinite list of real numbers = sequence

Ex: Fibonacci recursion

$$a_n = a_{n-1} + a_{n-2} \quad a_1 = a_2 = 1$$

$$a_3 = a_2 + a_1 = 1 + 1 = 2$$

$$a_4 = a_3 + a_2 = 2 + 1 = 3$$

A sequence $\{a_n\}$ can have a limit, written as

$$\lim_{n \rightarrow \infty} a_n = L \quad (a_n \xrightarrow{n \rightarrow \infty} L; a_n \rightarrow L \text{ as } n \rightarrow \infty)$$

If this limit exists as a finite number, we say $\{a_n\}$ converges; else diverges.

Ex: $\{a_n\} = \{(-1)^n\}$ $\lim_{n \rightarrow \infty} a_n$ Does not exist
sequence $\{a_n\}$ is divergent.

Ex: $\{a_n\}$ such that $a_n = \frac{n}{n+1}$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1 \quad (\text{convergent})$$

\downarrow
 0

Ex: $\{a_n\}$ such that $a_n = \frac{1}{n^k}$

$$k > 0 \quad \lim_{n \rightarrow \infty} \frac{1}{n^k} = 0 \quad (\text{convergent})$$

$$k < 0 \quad \lim_{n \rightarrow \infty} \frac{1}{n^k} = \lim_{n \rightarrow \infty} \frac{1}{n^{-|k|}} = \lim_{n \rightarrow \infty} n^{|k|} = \infty \quad (\text{divergent})$$

$$k = 0 \quad \lim_{n \rightarrow \infty} \frac{1}{n^0} = \lim_{n \rightarrow \infty} 1 = 1 \quad (\text{convergent}).$$

Remark: Use function $f(x)$ to define a sequence
 $a_n = f(n)$ for $n = 1, 2, \dots$

Limit-Rules

If $\{a_n\}$ and $\{b_n\}$ converge, then

$$1) \lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$2) \lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

$$3) \lim_{n \rightarrow \infty} (a_n \cdot b_n) = (\lim_{n \rightarrow \infty} a_n) \cdot (\lim_{n \rightarrow \infty} b_n)$$

$$4) \lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \quad \left(\lim_{n \rightarrow \infty} b_n \neq 0 \right)$$

$$5) \lim_{n \rightarrow \infty} (c \cdot a_n) = c \lim_{n \rightarrow \infty} a_n$$

6) If f is continuous then

$$\lim_{n \rightarrow \infty} f(a_n) = f(\lim_{n \rightarrow \infty} a_n)$$

Ex: $f(x) = \arctan(\sqrt{x})$

$$a_n = \frac{1}{n}, \quad n=1, 2, \dots$$

$$\lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} \arctan(\sqrt{a_n})$$

To use (6), we need to check that $\lim_{n \rightarrow \infty} a_n$ exists

and f is continuous.

- $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ ✓

- f continuous ✓ because \arctan is continuous and $\sqrt{}$ is continuous (for non-negative values).

$$\lim_{n \rightarrow \infty} \arctan(\sqrt{a_n}) \stackrel{\text{arctan is continuous}}{\Rightarrow} \arctan(\lim_{n \rightarrow \infty} \sqrt{a_n}) \stackrel{\sqrt{} \text{ is continuous}}{\Rightarrow} \arctan(\sqrt{\lim_{n \rightarrow \infty} a_n})$$

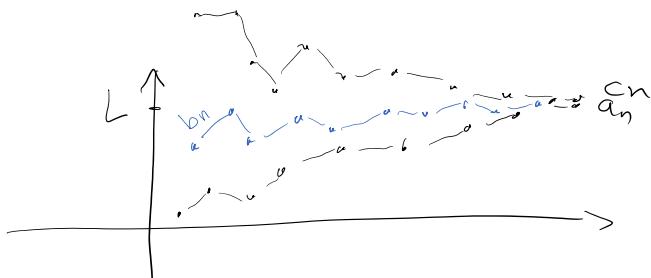
$$= \arctan(\sqrt{\lim_{n \rightarrow \infty} \frac{1}{n}}) = \arctan(\sqrt{0}) = \arctan(0) = 0$$

Squeeze Theorem

If $a_n \leq b_n \leq c_n$ for all $n \geq N$ [for some large N]

and $\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$

then $\lim_{n \rightarrow \infty} b_n = L$



Ex: $b_n = \frac{n!}{n^n}$ Find $\lim_{n \rightarrow \infty} b_n$

$$b_n = \frac{n!}{n^n} = \frac{n \cdot [n-1] \cdot [n-2] \cdot [n-3] \cdots [2] \cdot [1]}{n \cdot n \cdot n \cdot n \cdots n \cdot n} \leq \frac{1}{n}$$

$$\underbrace{a_n = 0}_{a_1=0, a_2=0, \dots} \leq b_n = \frac{n!}{n^n} \leq \frac{1}{n} = c_n \quad < 1$$

$$a_1=0, a_2=0,$$

$$a_3=0, a_4=0,$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} 0 = 0 = \lim_{n \rightarrow \infty} \frac{1}{n} = \lim_{n \rightarrow \infty} c_n$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$$

Monotone sequences:

$\{a_n\}$ is monotone increasing if $a_n \leq a_{n+1} \leq a_{n+2} \leq \dots$

$\{a_n\}$ is monotone decreasing if $a_n \geq a_{n+1} \geq a_{n+2} \geq \dots$

$\{a_n\}$ is bounded from above if it exists M s.t.

$a_n \leq M$ for all n .

$\{a_n\}$ is bounded from below if it exists M s.t.

$M \leq a_n$ for all n .

Basic Property:

If $\{a_n\}$ is monotone increasing and bounded from above, then $\{a_n\}$ converges.

If $\{a_n\}$ is monotone decreasing and bounded from below, then $\{a_n\}$ converges.

