M1ZB3 Lecture 35 Part 1 (C02) Dr. Wolkowicz
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§ 14.6 cont'd The Gradient Vector

Directional derivative of $f(x,y)$ in the direction of the unit vector $u = \langle a,b \rangle$ is

$$D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b.$$

$$= \underbrace{\langle f_x(x,y), f_y(x,y) \rangle}_{\text{called the gradient vector.}} \cdot \langle a,b \rangle$$

Def'n The GRADIENT of $f(x,y)$ is denoted

grad f

or ∇f (nabla f)

or del f

and $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle.$

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$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

Example : $f(x,y) = \cos(x) + x^2 y$

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$
$$= \langle -\sin(x) + 2xy, \quad x^2 \rangle$$

NOTE: $\vec{u} = \langle a, b \rangle$

$\|\vec{u}\| = 1$

$$D_{\vec{u}} f(x,y) = f_x a + f_y b$$

$$= \langle f_x, f_y \rangle \cdot \langle a, b \rangle$$
$$= \nabla f(x,y) \cdot \langle a, b \rangle$$
$$= \nabla f(x,y) \cdot \vec{u}$$

This is the scalar projection of the gradient vector onto $\vec{u}$.

Gradient vector in (3 dimensions)

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$f(x, y, z)$

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$$

$\|\vec{u}\| = 1$

Maximizing the Directional Derivative.

① In what direction does f change the fastest, i.e. what is the direction of steepest ascent, and what is the direction of steepest descent?

② What is the maximum rate of change of f?

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Thm/ Let f be a differentiable function of 2 or 3 variables. The maximum value of the directional derivative $D_{\vec{u}} f$ $(\|u\|=1)$ is $\|\nabla f\|$

and it occurs when $\vec{u}$ has the SAME direction as ∇f.

Proof: $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$$= \|\nabla f\| \underbrace{\|\vec{u}\|}_{=1} \cos(\theta)$$

$= \|\nabla f\| \cos(\theta)$. where θ is the angle between ∇f and $\vec{u}$.

The maximum occurs when $\cos(\theta) = 1$, i.e. $\theta = 0$.

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Hence, the max. value of $D_{\vec{u}} f$ is $\|\nabla f\|$, and it occurs when $\vec{u}$ is in the same direction as ∇f.

$$\therefore \vec{u} = \frac{1}{\|\nabla f\|} \nabla f$$