

April 2, 2020

March 25, 2020 8:30 PM

## § 14.6 cont'd Directional Derivatives

Thm If  $f$  is a differentiable function of  $x$  and  $y$ , then  $f$  has a directional derivative in the direction of an unit vector  $\vec{u} = \langle a, b \rangle$ , and

$$D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Proof: Let  $\vec{u} = \langle a, b \rangle$ , a unit vector, and define a 1-variable function

$$\begin{aligned} g(z) &= f((x_0+y_0) + z\vec{u}) \\ &= f(x_0 + za, y_0 + zb) \end{aligned}$$

By the definition of derivative

$$\begin{aligned}
 g'(0) &= \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\
 &= D_u^{\perp} f(x_0, y_0) \quad \text{⊗}
 \end{aligned}$$

Alternatively, use the  
2-variable chain rule

$$\begin{aligned}
 g'(0) &= f_x(x_0, y_0) \frac{d}{dt} (x_0 + ta) \Big|_{t=0} \\
 &\quad + f_y(x_0, y_0) \frac{d}{dt} (y_0 + tb) \Big|_{t=0} \\
 &= f_x(x_0, y_0) a + f_y(x_0, y_0) b \\
 D_u^{\perp} f(x_0, y_0) &\stackrel{\text{⊗}}{=} g'(0) \stackrel{\text{⊗}}{=} f_x(x_0, y_0) a + f_y(x_0, y_0) b. \\
 \therefore D_u^{\perp} f(x_0, y_0) &= f_x(x_0, y_0) a + f_y(x_0, y_0) b. \quad \text{x.}
 \end{aligned}$$

# What about 3 - variables?

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$$\vec{u} = \langle a, b, c \rangle$$

$$\|\vec{u}\| = (a^2 + b^2 + c^2)^{1/2} = 1.$$

$$D_{\vec{u}} f(x_0, y_0, z_0)$$

$$= \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h}$$

$$= f_x(x_0, y_0, z_0) a + f_y(x_0, y_0, z_0) b + f_z(x_0, y_0, z_0) c.$$

Example. Find the directional derivative of

$$f(x, y, z) = \underline{x^2 y + y^2 z}$$

at  $(1, 2, 3)$

in the direction of the unit vector in the direction of  $\vec{v} = \langle 2, -1, 2 \rangle$ .

$$\text{Find } \|\vec{v}\| = (z^2 + (-1)^2 + (2^2))^{1/2} \\ = 3$$

The unit vector in the direction of  $\vec{v}$ , is

$$\hat{u} = \frac{1}{\|\vec{v}\|} \vec{v} = \frac{1}{3} \langle 2, -1, 2 \rangle \\ = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle.$$

$$\begin{aligned} D_{\vec{u}} f(1, 2, 3) &= f_x(1, 2, 3) \left(\frac{2}{3}\right) + f_y(1, 2, 3) \left(-\frac{1}{3}\right) \\ &\quad + f_z(1, 2, 3) \left(\frac{2}{3}\right) \\ &= 2xy \Big|_{(1, 2, 3)} \left(\frac{2}{3}\right) + (x^2 + 2yz) \Big|_{(1, 2, 3)} \left(-\frac{1}{3}\right) \\ &\quad + y^2 \Big|_{(1, 2, 3)} \left(\frac{2}{3}\right) \\ &= (2)(1)(2)\left(\frac{2}{3}\right) + (1^2 + 2(2)(3))\left(-\frac{1}{3}\right) + 2^2\left(\frac{2}{3}\right) \\ &= 1 \end{aligned}$$