M1ZB3 Lecture 34 Part 1 (C02) Dr. Wolkowicz
April 2, 2020

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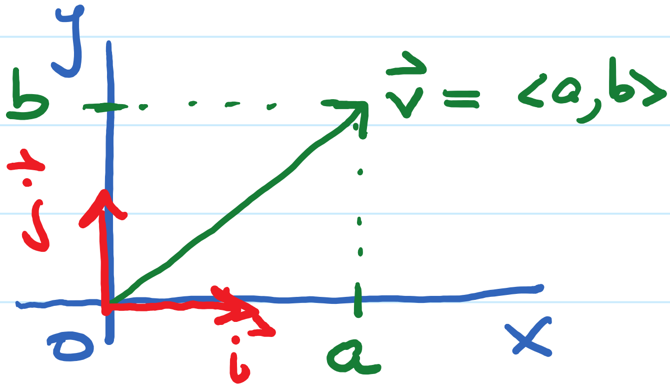
§14.6 Directional Derivatives & the Gradient Vector

Vectors in $\mathbb{R}^2$



Write
$\vec{v} = \underbrace{\langle a, b \rangle}$
components of v.

$\vec{i} = \langle 1, 0 \rangle$ unit vector in direction of pos. x-axis

$\vec{j} = \langle 0, 1 \rangle$
Unit vector
in the direction of the
positive y-axis.

$\vec{v} = a\vec{i} + b\vec{j}$

The LENGTH of $\vec{v} = \langle a, b \rangle$

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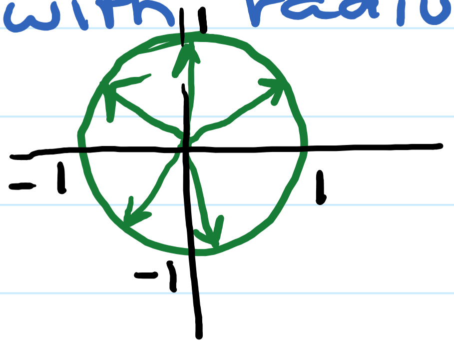
denoted $\|\vec{v}\| = (a^2 + b^2)^{1/2}$

$\vec{v}$ is a <u>unit</u> vector if $\|\vec{v}\| = 1$.

$\Rightarrow$ head of $\vec{v}$ lies on the unit circle

or circle with radius 1.

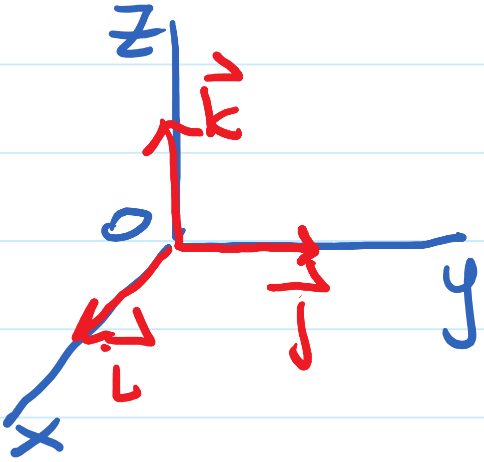
unit vectors.



$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$$

is a unit vector in the direction of $\vec{v}$.

Vectors in $\mathbb{R}^3$



$\vec{i} = \langle 1, 0, 0 \rangle$

$\vec{j} = \langle 0, 1, 0 \rangle$

$\vec{k} = \langle 0, 0, 1 \rangle$

$$\vec{v} = \langle a, b, c \rangle$$

$$\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$$

Length of $\vec{v}$ is

$$\|\vec{v}\| = (a^2+b^2+c^2)^{1/2}$$

$\vec{v}$ is a unit vector if

$$\|\vec{v}\| = 1.$$

Properties of vectors

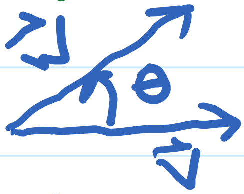
$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$\vec{w} = \langle w_1, w_2, w_3 \rangle$$

(i) $\vec{v} \pm \vec{w} = \langle v_1 \pm w_1, v_2 \pm w_2, v_3 \pm w_3 \rangle$

(ii) $\alpha \in \mathbb{R}$, $\alpha\vec{v} = \langle \alpha v_1, \alpha v_2, \alpha v_2 \rangle$.

Dot product: $\vec{v} \cdot \vec{w}$



$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$= \|\vec{v}\| \, \|w\| \cos\theta$$

RECALL: If $z = f(x,y)$

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$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

$$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}.$$

$f_x(x_0,y_0)$ is the rate of change of z at (x_0,y_0) in the direction of $\vec{i} = \langle 1,0\rangle$

$f_y(x_0,y_0)$ is the rate of change of z at (x_0,y_0) in the direction $\vec{j} = \langle 0,1\rangle$.

Question: What is the rate of change in the direction of a unit vector $\vec{u}$?

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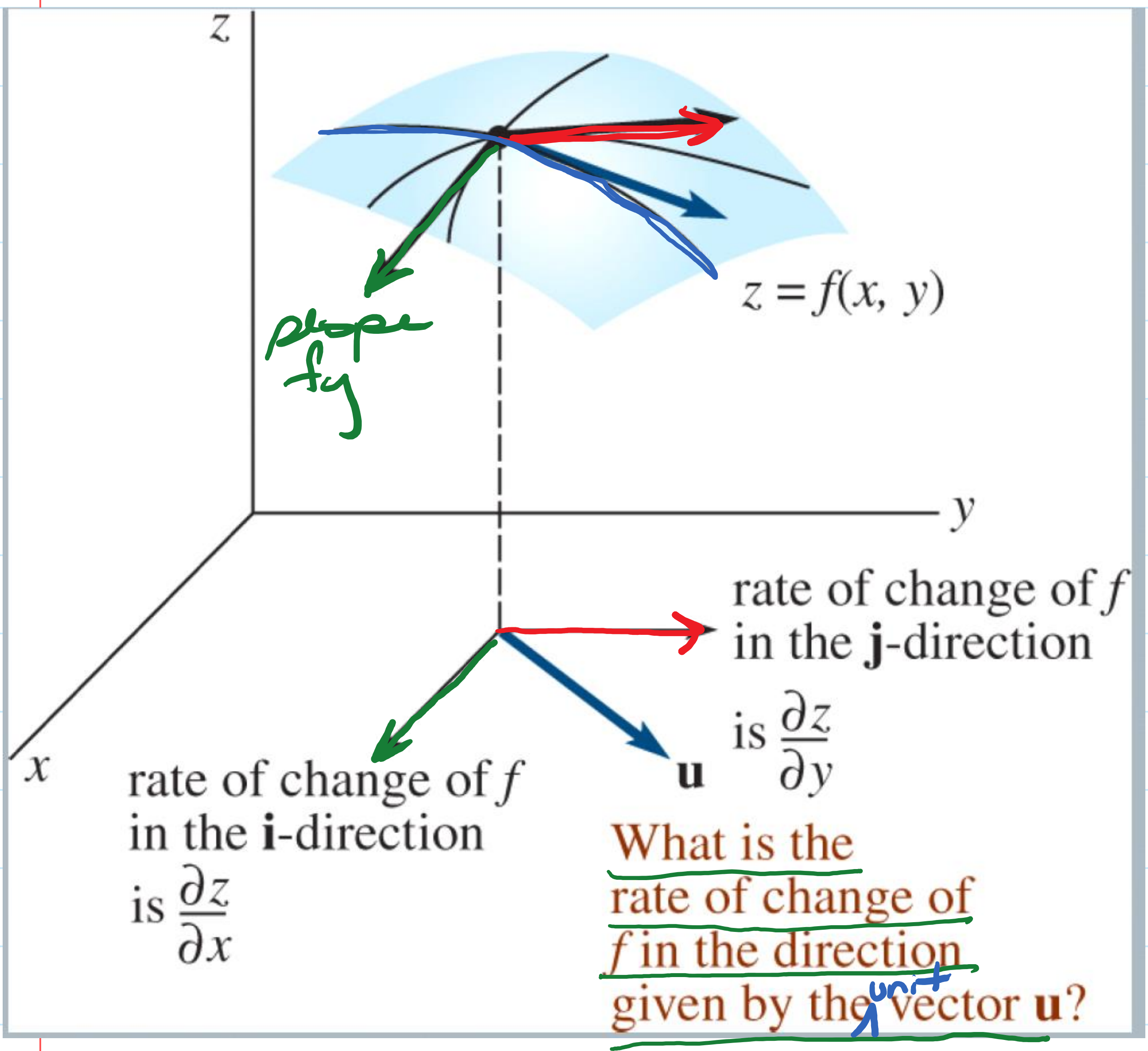
What is the rate of change of z in the direction of the unit vector $\vec{u} = \langle a, b \rangle$, called the directional derivative?



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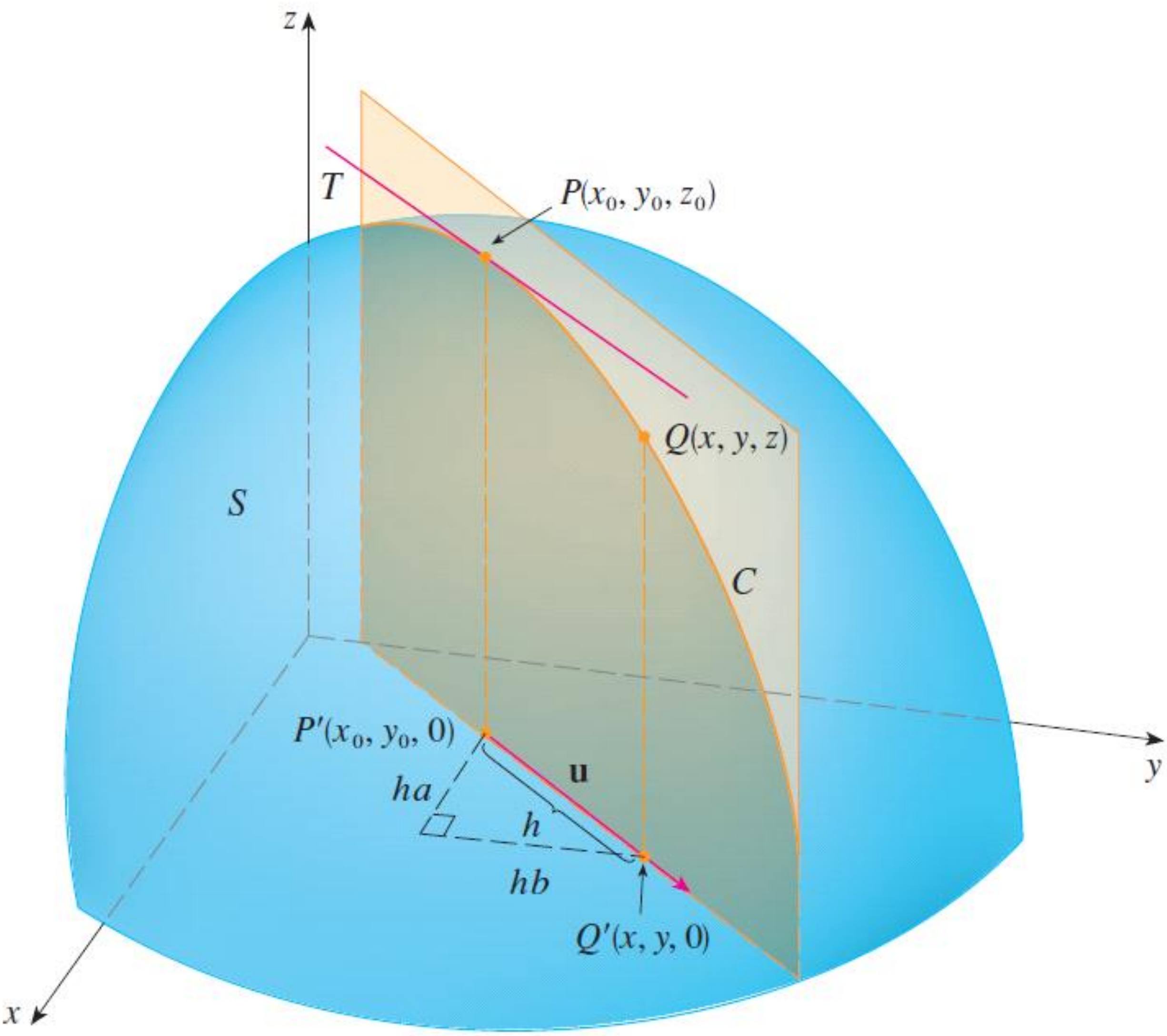
Slope of T, the tangent line in the direction of the unit vector $\vec{u}$ is the directional derivative, denoted

$$D_{\vec{u}} f(x_0, y_0)$$

where $\|\vec{u}\| = 1$.



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The DIRECTIONAL DERIVATIVE of $f(x,y)$ at (x_0, y_0) in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ is:

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f((x_0,y_0) + h\vec{u}) - f(x_0,y_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h},$$

if this limit exists.

$$\boxed{D_{u} f(x_0, y_0) = a f_x(x_0, y_0) + b f_y(x_0, y_0)}$$

NOTE:

$$D_{\vec{i}} f(x_0, y_0) = f_x(x_0, y_0), \quad \vec{i} = \langle 1, 0 \rangle$$

$$D_{\vec{j}} f(x_0, y_0) = f_y(x_0, y_0), \quad \vec{j} = \langle 0, 1 \rangle.$$

Example.

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Find the directional derivative of $f(x,y) = e^x \sin(y)$ at $(0, \frac{\pi}{3})$ in the direction

of the unit vector in the direction of $\vec{v} = \langle -6, 8 \rangle$.

Sol'n: Find a unit vector in the direction of $\vec{v}$.

$\|\vec{v}\| = ((-6)^2 + (8)^2)^{1/2}$

$= (36 + 64)^{1/2} = (100)^{1/2}$

$= 10.$

unit vector in direction of $\vec{v}$ is:

$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v} = \langle -\frac{6}{10}, \frac{8}{10} \rangle.$

$D_{\vec{u}} f(0, \frac{\pi}{3})$

$= f_x(0, \frac{\pi}{3})(-\frac{6}{10}) + f_y(0, \frac{\pi}{3})(\frac{8}{10})$

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$$= \left(e^x \sin(y)\Big|_{(x,y)=(0,\frac{\pi}{3})}\right)\left(-\frac{6}{10}\right) + \left(e^x \cos(y)\Big|_{(x,y)=(0,\frac{\pi}{3})}\right)\left(\frac{8}{10}\right)$$

$$= -\frac{6}{10}\,\frac{\sqrt{3}}{2} + \left(\frac{1}{2}\right)\left(\frac{8}{10}\right)$$

$$= \frac{-3\sqrt{3}+4}{10}.$$

$D_u f(0, \frac{\pi}{3})$