M1ZB3 Lecture 33 Part 2 (C02) Dr. Wolkowicz
March 31

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§14.5. Implicit Differentiation (with the Chain RULE)

Assume $F(x,y) = 0$ defines y implicitly as a function of x.

i.e., $y = f(x)$ and $F(x, f(x)) = 0$ for all x in the domain of f.

If F is differentiable, by Case 1 of Chain Rule, differentiate both sides wrt x:

$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

If $\frac{\partial F}{\partial y} \neq 0$,

solve for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y} = -F_x/F_y .$$

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Example.

$$F(x,y) = y^5 - 3x^2y^3 + x^5 - 1 = 0$$

Find $\frac{dy}{dx}$.

Sol'n:

$$\frac{dy}{dx} = \frac{-F_x}{F_y}, \text{ provided } F_y \neq 0.$$

$$= -\frac{(-6xy^3 + 5x^4)}{5y^4 - 9x^2y^2}$$

$$= \frac{x(6y^3 - 5x^3)}{y^2(5y^2 - 9x^2)}, \quad y \neq 0, \; 5y^2 \neq 9x^2.$$

Question: Given $F(x,y) = 0$. When do you know there is a function $y(x)$ so that $F(x, y(x)) = 0$ and $\frac{dy}{dx}$ exists?

Implicit Function Theorem (IFT)

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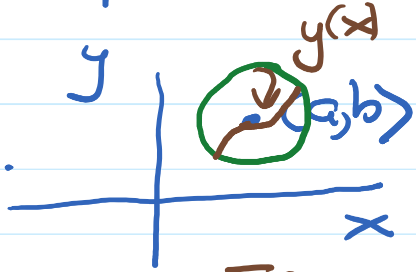
(simplest).

If F is defined on a disk containing (a,b) where $F(a,b)=0$, $F_y(a,b)\neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x,y)=0$ defines y as a function of x near the point (a,b) and

$$\boxed{\frac{dy}{dx} = \frac{-F_x}{F_y}}.$$



$F(x, y(x))=0$.

Previous Example: Let $(a,b)=(0,1)$.

Then $F(0,1) = 1-0+0-1 = 0$.

F_x and F_y are continuous

$F_y(0,1) = 1(5(1)-0) = 5 \neq 0$.

Then by the (IFT) there is a function $y(x)$ so that

$$(y(x))^5 - 3x^2(y(x))^3 + x^5 - 1 = 0.$$

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What if F is a function of 3 variables, x, y, z and $z = z(x,y)$.

Assume $z = z(x,y)$ Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

given $F(x,y,z) = c$, constant.

$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{dx}}_{=0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial c}{\partial x} = 0.$$

$$\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z} = -\frac{F_x}{F_z}$$

provided $F_z \neq 0$.

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}, \text{ provided } F_z \neq 0.$$

Example.

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$F(x,y,z) = \sin(x+y) + \sin(x+z) + \sin(y+z) = 0.$

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

Sol'n.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \text{ provided } F_y \neq 0$$

$$= -\frac{(\cos(x+y) + \cos(x+z) + 0)}{(0 + \cos(x+z) + \cos(y+z))}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$= -\frac{(\cos(x+y) + 0 + \cos(y+z))}{(0 + \cos(x+z) + \cos(y+z))}$$

Implicit Function Thm.

Assume $F(x,y,z)$ is a function of 3 variables,

$F(a,b,c) = C.$

$F_z(a,b,c) \neq 0$

As well F_x, F_y, and F_z are continuous near (a, b, c),

Then there exists a differentiable function

$$z = z(x, y),$$

defined near (a, b) such that

$$F(x, y, z(x,y)) = K$$

for all (x, y) near (a, b),

and z is differentiable with partial derivatives,

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \; ; \; \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}.$$