M1ZB3 Lecture 33 Part 1 (C02) Dr. Wolkowicz
March 31

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Note that I am assuming that the lecture on Friday, March 27 was cancelled, since you were writing your term test all day. Therefore, on Tuesday, March 31, you will be welcome to ask questions about any of the on-line lectures, but we will start with questions on Lecture 32. Once all questions are answered, we will work on problems together.

I am not expecting that you listened to this lecture before class on Tuesday, March 31, and so we will begin the lecture on Thursday with questions about this lecture. However, if you have listened to this lecture and wish to ask questions on Tuesday, March 31, that will be fine.

§14.5 Chain Rule cont'd.

If $z(x, y, w)$ and $x(s,t)$
$y(s,t)$
$z(s,t)$

then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial t}$$

Matrix Notation

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$$\underbrace{\begin{bmatrix} \frac{\partial z}{\partial s} \end{bmatrix}}_{1\times 1} = \underbrace{\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial w} \end{bmatrix}}_{1\times 3} \underbrace{\begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \\ \frac{\partial w}{\partial s} \end{bmatrix}}_{3\times 1}$$

$$\frac{\partial z}{\partial t} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial w} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial w}{\partial t} \end{bmatrix}$$

AND.

$$\underbrace{\begin{bmatrix} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix}}_{1\times 2} = \underbrace{\underbrace{\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial y} \end{bmatrix}}_{1\times 3} \underbrace{\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial w}{\partial s} & \frac{\partial w}{\partial t} \end{bmatrix}}_{3\times 2}}_{1\times 2}.$$

Example.

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$$z = f(x,y), \quad x = r^2 + s^2$$
$$y = 2rs$$

Find $\dfrac{\partial^2 z}{\partial r \partial x}$

Sol'n: $\dfrac{\partial x}{\partial r} = 2r \qquad \dfrac{\partial y}{\partial r} = 2s$

$$\frac{\partial^2 z}{\partial r \partial x} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial r}{\partial y}$$

$$= \frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y \partial x}(2s)$$

Chain Rule (many variable case.

Assume $u(x_1, x_2, \dots, x_n)$, a function of n-variables. and each $x_i(t_1, t_2, \dots, t_m)$ is a function of m-variables.

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$$\frac{\partial u}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial t_j}, \quad j = 1, 2, \ldots, m$$

Example.

$$w = xy + yz + zx$$

where

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta.$$

Find. $\dfrac{\partial w}{\partial r}$. $\quad (\underset{x,y,z}{n=3}, \underset{r,\theta}{m=2})$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= (y+z)\cos\theta + (x+z)\sin\theta + (y+x)\theta$$

$$= (r\sin\theta + r\theta)\cos\theta + (r\cos\theta + r\theta)\sin\theta + (r\sin\theta + r\cos\theta)\theta.$$

HW. TRY $\dfrac{\partial w}{\partial \theta}$.

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Matrix Formulation.

$u(x_1, x_2, \dots, x_n)$

$x_j(t_1, t_2, \dots, t_m) \quad j = 1, \dots, n.$

$$\frac{\partial u}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial t_j}.$$

$$\left[\frac{\partial u}{\partial t_1}, \frac{\partial u}{\partial t_2}, \dots, \frac{\partial u}{\partial t_m}\right]$$

$1 \times m$

$$= \underbrace{\underbrace{\left[\frac{\partial u}{\partial x_1} \ \frac{\partial u}{\partial x_2} \ \cdots \ \frac{\partial u}{\partial x_n}\right]}_{1 \times n} \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_m} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_m} \end{bmatrix}}_{n \times m}}_{1 \times m.}$$