

March 26

March 20, 2020 7:50 PM

# §14.4 cont'd

## Differentials for functions of $n$ -variables.

$$W = f(x_1, x_2, \dots, x_n)$$

TOTAL DERIVATIVE

$$dW = \sum_{i=1}^n f_{x_i} dx_i$$

Example:  $W = \sum_{i=1}^n x_i^2 = x_1^2 + x_2^2 + \dots + x_n^2$

$$dx_i = 2x_i$$

$$dW = \sum_{i=1}^n 2x_i dx_i$$

$$= 2x_1 dx_1 + 2x_2 dx_2 + \dots + 2x_n dx_n$$

Actual difference

$$\Delta W = f(x_1 + \Delta x_1, \dots, x_n + \Delta x_n) - f(x_1, x_2, \dots, x_n)$$

$$\Delta W \approx dW$$

for  $dx_j \approx \Delta x_j$   
 $\Delta x_j$  small,  $j=1, 2, \dots, n$ .

$$\Delta W \approx L(x_1 + \Delta x_1, \dots, x_n + \Delta x_n) - f(x_1, x_2, \dots, x_n) = \sum_{j=1}^n f_{x_j} \Delta x_j$$

# § 14.5 Chain Rule

1- variable

$$\frac{d}{dt} (F(g(t))) = f'(g(t)) g'(t)$$

OR if  $y = f(x)$ ,  $x = g(t)$

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

Example:

$$\begin{aligned} \frac{d}{dt} \cos(e^t) &= -\sin(e^t) \frac{d}{dt} e^t \\ &= -\sin(e^t) e^t \end{aligned}$$

2- variable chain rule.

Case 1:  $z = f(x, y)$

$x = g(t)$ ,  $y = h(t)$

(Require  $f(x, y)$  to be differentiable as a function of 2 variables.)

CHAIN RULE:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

## Example.

March 26, 2020

8:14 AM

$$z = f(x, y) = x^3 y^5$$

$$\text{where } x = \cos(t) \\ y = \sin(t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$= 3x^2 y^5 (-\sin(t)) + 5x^3 y^4 \cos(t)$$

$$= -3 \cos^2(t) \sin^6(t) \\ + 5 \cos^4(t) \sin^4(t).$$

Case 2.  $z = f(x, y)$   
 $x = g(s, t), y = h(s, t)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

Example.

Let  $f(s^2 - t^2, t^2 - s^2)$   
where  $f(x, y)$  is differentiable.

Show that  $f$  satisfies  
the PDE

$$t \frac{\partial u}{\partial s} + s \frac{\partial u}{\partial t} = 0$$

$$u = f(s^2 - t^2, t^2 - s^2)$$

$$\text{Let } x = s^2 - t^2$$

$$y = t^2 - s^2$$

$$\begin{aligned} \textcircled{1} \quad \frac{\partial u}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \\ &= f_x(2s) + f_y(-2s^2) \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad \frac{\partial u}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \\ &= f_x(-2t) + f_y(2t) \end{aligned}$$

$$t \times \textcircled{1} + s \times \textcircled{2}$$

$$\cancel{2tsf_x} - \cancel{2tsf_y} \sim \cancel{2tsf_x} + \cancel{2tsf_y}$$

$$= 0$$