M1ZB3 Lecture 32 Part 1 (C02) Dr. Wolkowicz
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§14.4 Linear Approximations

Consider function $f(x,y)$ with (a,b) in its domain.

RECALL: Tangent plane to $f(x,y)$ at (a,b) is

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b).$$

Def'n. The "linearization" or "linear approximation" or "tangent plane approx." of f at (a,b) is

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b).$$

IDEA: Just as the tangent line to a function $y = g(x)$ (a function of 1-variable) is an approximation to the function near a point x_0

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The tangent plane at (a,b) is an approximation to $z = f(x,y)$ near (a,b)

Example: Estimate $f(3.01, 4.02)$ using the linear approximation where $f(x,y) = (x^2+y^2)^{1/2}$

Sol'n: LINEARIZE at $(a,b) = (3,4)$

$f(3,4) = (3^2+4^2)^{1/2} = 5$.

$$f_x = \frac{1}{2}(x^2+y^2)^{-1/2}\, 2x = \frac{3}{5} = 0.6$$

$$f_y = \frac{1}{2}(x^2+y^2)^{-1/2}\, 2y = \frac{4}{5} = 0.8$$

$$L(x,y) = 0.6(x-3) + 0.8(y-4) + 5$$

$f(x,y) \approx L(x,y)$ near $(3,4)$

$$L(3.01, 4.02) = .6(.01) + 0.8(.02) + 5 = 5.022.$$

Using a calculator

$f(3.01, 4.02) = 5.0220015\overline{9}$.

L approximates f to 5 decimals.

fcns of n-variables

If $f(x_1, x_2, \ldots, x_n)$ is a function of n-variables, the linearization of f at $(a_1, a_2, \ldots, a_n)$ in the domain of f is

$$L(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} f_{x_j}(a_1, a_2, \ldots, a_n)(x_j - a_j) + f(a_1, a_2, \ldots, a_n)$$

$w = L(x_1, x_2, \ldots, x_n)$ is the equation of the "tangent hyperplane" to f at $(a_1, a_2, \ldots, a_n)$

Thm/ If $z = f(x,y)$ and f_x and f_y exist near (a,b) and are continuous at (a,b), then f is <u>differentiable at (a,b)</u>.

Then the linear approximation is a good approximation near (a,b).

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Differentials.

$y = f(x)$, $\frac{dy}{dx} = f'(x)$.

$$dy = f'(x)\,dx$$

$\frac{dy}{dx}$ is a differential of $\frac{y}{x}$.

dy & dx are viewed as independent variables.

$z = f(x,y)$

dz differential of z (also use df)

TOTAL DIFFERENTIAL.

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$\shortparallel$

df

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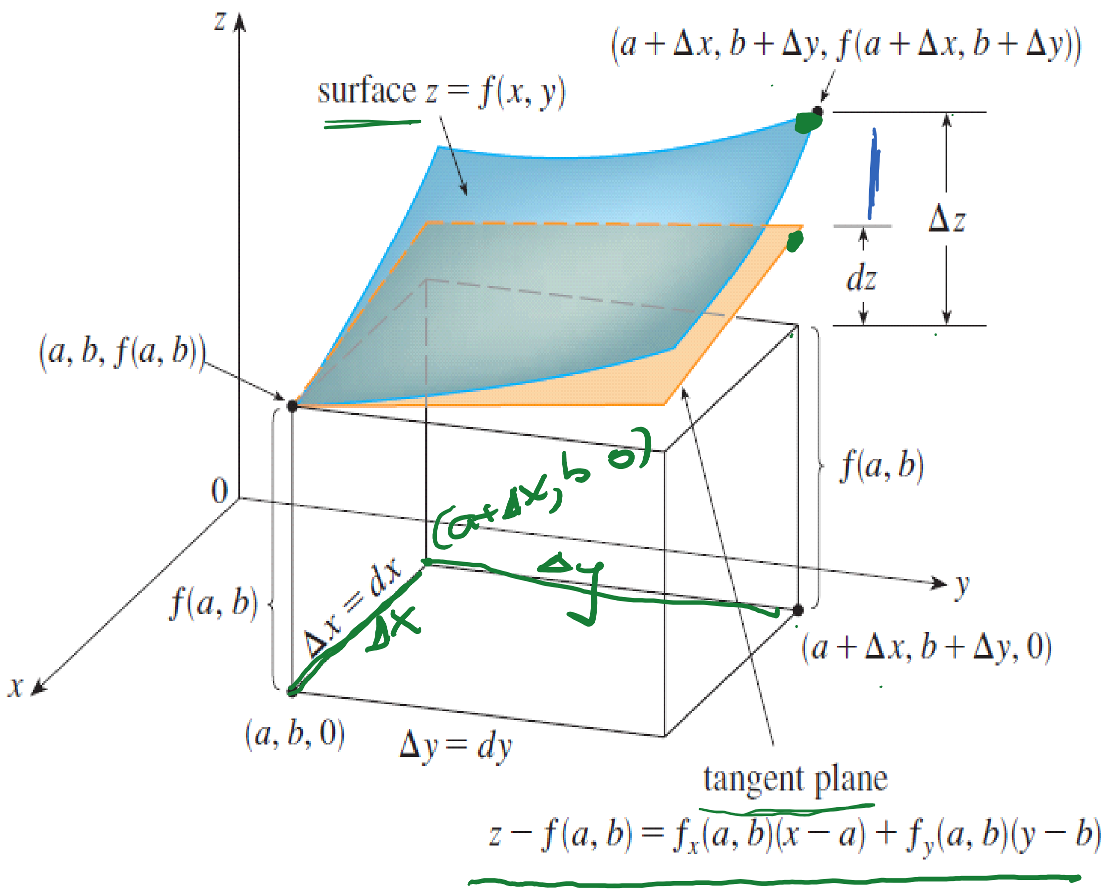
Example. One leg of a triangle increases from 3cm to 3.2cm. while the other leg decreases from 4cm to 3.96 cm. Use a total differential to approximate the change in the length of the hypotenuse.



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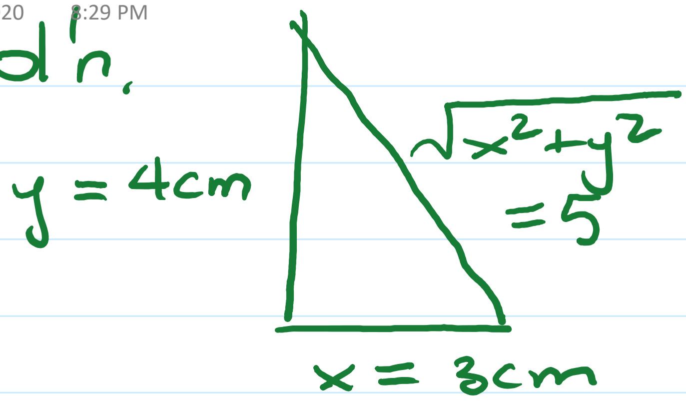
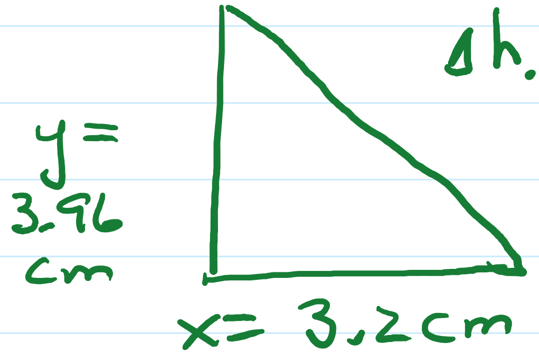
Sol'n.





$$h(x,y) = \sqrt{x^2+y^2}$$

$$x = 3 \qquad \Delta x = dx = .2$$

$$y = 4 \qquad \Delta y = dy = -0.04.$$

$$dh = h_x dx + h_y dy$$

$$= \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cancel{2}x \, dx$$

$$+ \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cancel{2}y \, dy$$

$$= \frac{1}{5}(3)(.2) + \frac{1}{5}(4)(-.04)$$

$$= .6/5 - .16/5 = \frac{.44}{5} = .088 cm.$$

change creates an increase in the hypotenuse of 0.088 cm

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$h(3.2, 3.96) - h(3,4)$

$= 5.091326 - 5$

$= 0.091326$.

Our approx of the change in the hypotenuse, 0.088 cm is good to 2 decimals.