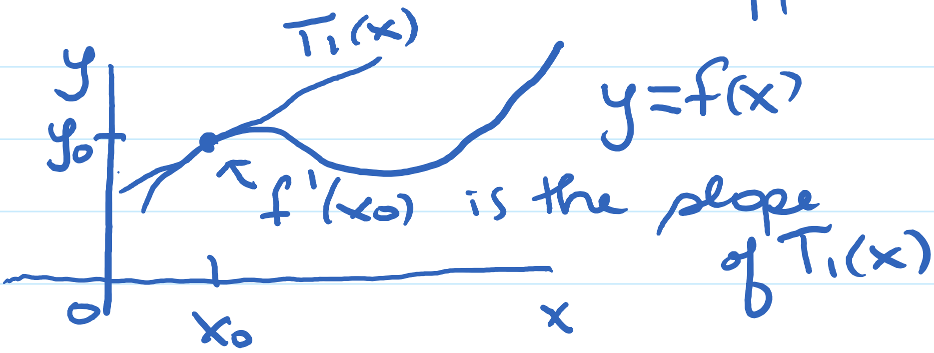
M1ZB3 Lecture 31 Part 2 (C02) Dr. Wolkowicz
March 24

March 20, 2020 7:47 PM

§14.4 Tangent Planes & Linear Approximations



The equation of a tangent line to a curve $y=f(x)$ at the point (x_0, y_0) has slope $f'(x_0)$ and is given by

$$y = f(x_0) + f'(x_0)(x-x_0).$$

$$= T_1(x) \quad \text{(Taylor polynomial)}$$

Let $P(x_0, y_0, z_0)$ be a point in space.

A plane passing through P has an equation of the form

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$$

(A, B, C not ALL be zero.)

If the plane is NOT parallel to the z axis, $C \neq 0$, and

March 23, 2020 7:51 PM

we can polve for z.

$$z = a(x-x_0) + b(y-y_0) + z_0$$

where

$$a = -A/C, \quad b = B/C$$

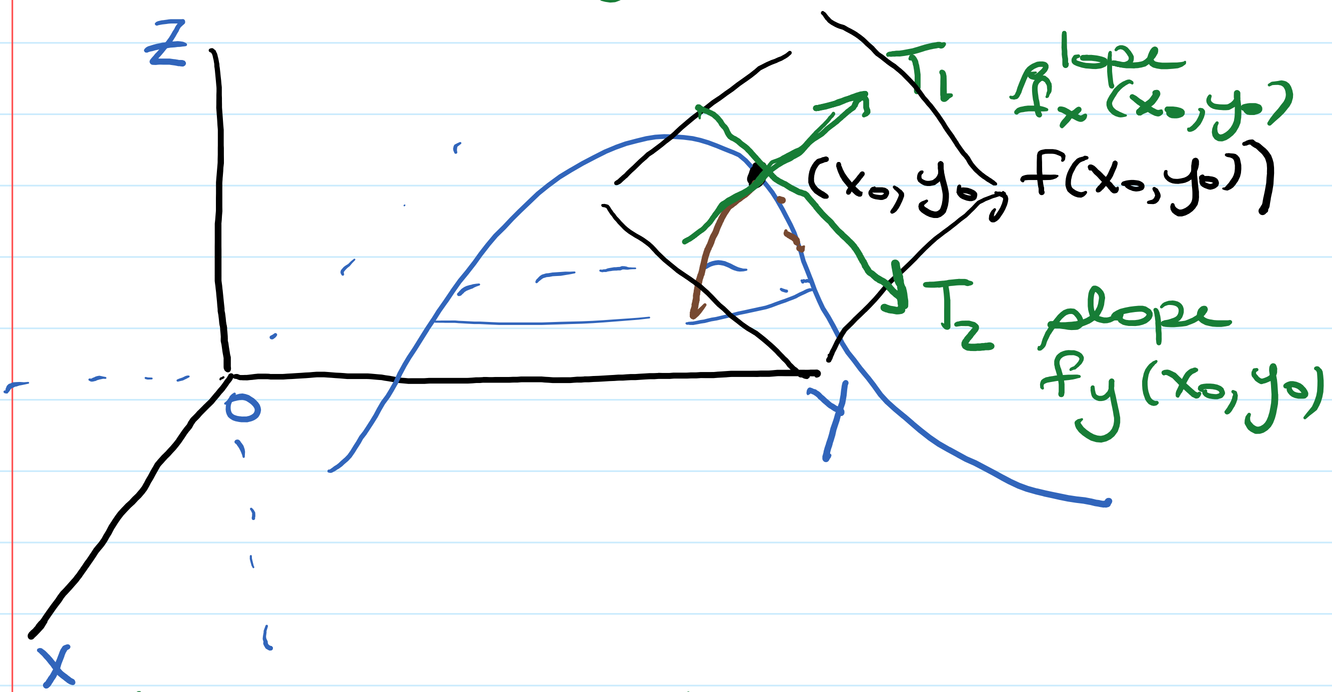
Note:

$\frac{\partial z}{\partial x} = a =$ plope of the xz-plice.

$\frac{\partial z}{\partial y} = b =$ slope of the yz-plice.

Consider the surface (i.e. the graph) of

$$z = f(x, y)$$



Let (x_0, y_0, z_0) be a point on the surface.

Slice parallel to the xz plane

through $(0, y_0, 0)$.

March 23, 2020 7:51 PM

We see

a = slope of

$g(x) = f(x, y_0)$ at x_0

$g'(x_0) = f_x(x_0, y_0)$

= slope of $T_1(x)$

the slope of the tangent plane at $(x_0, y_0, f(x_0, y_0))$

Similarly,

$b = f_y(x_0, y_0)$ the slope of T_2

EQUATION of the TANGENT PLANE to $z = f(x,y)$ at $(x_0, y_0, f(x_0, y_0))$ is

$$\boxed{\begin{aligned} z = {} & f_x(x_0, y_0)(x - x_0) \\ & + f_y(x_0, y_0)(y - y_0) \\ & + f(x_0, y_0) \end{aligned}}$$

Example. Find the tangent plane to $z = 2x^2 - y^3$ at $(3, 1, 17)$.

NOTE: $2(3)^2 - 1^3 = 17$

$\uparrow x_0$ $\quad \uparrow y_0 \quad \uparrow z_0 = f(x_0, y_0)$

March 24, 2020 6:59 AM

Point <u>must</u> be on the surface.

$$f(x,y) = 2x^2 - y^3$$

$f_x = 4x$ $\qquad$ $f_y = -3y^2$

$f_x(3,1) = 12$ $\qquad$ $f_y(3,1) = -3$

$$z = \underbrace{12}_{f_x(x_0,y_0)}(x - \underbrace{3}_{x_0}) + \underbrace{(-3)}_{f_y(x_0,y_0)}(y - \underbrace{1}_{y_0}) + \underbrace{17}_{z_0}.$$

(Check $(3, 1, 17)$ is on the plane).

BEWARE: Badly behaved function (no tangent plane at a particular point)

EXAMPLE.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

March 23, 2020 7:51 PM

Maple Program.

$$f(x,y) := ifelse\left(x=0 \text{ and } y=0,\ 0,\ \frac{x\cdot y}{(x^2+y^2)}\right);$$

$$f := (x,y) \mapsto ifelse\left(x=0 \text{ and } y=0,\ 0,\ \frac{x\,y}{y^2+x^2}\right)$$

$plot3d(f(x,y), x=-.25\,...25, y=-.25\,...25);$

This graph is very "jagged" near (0,0).

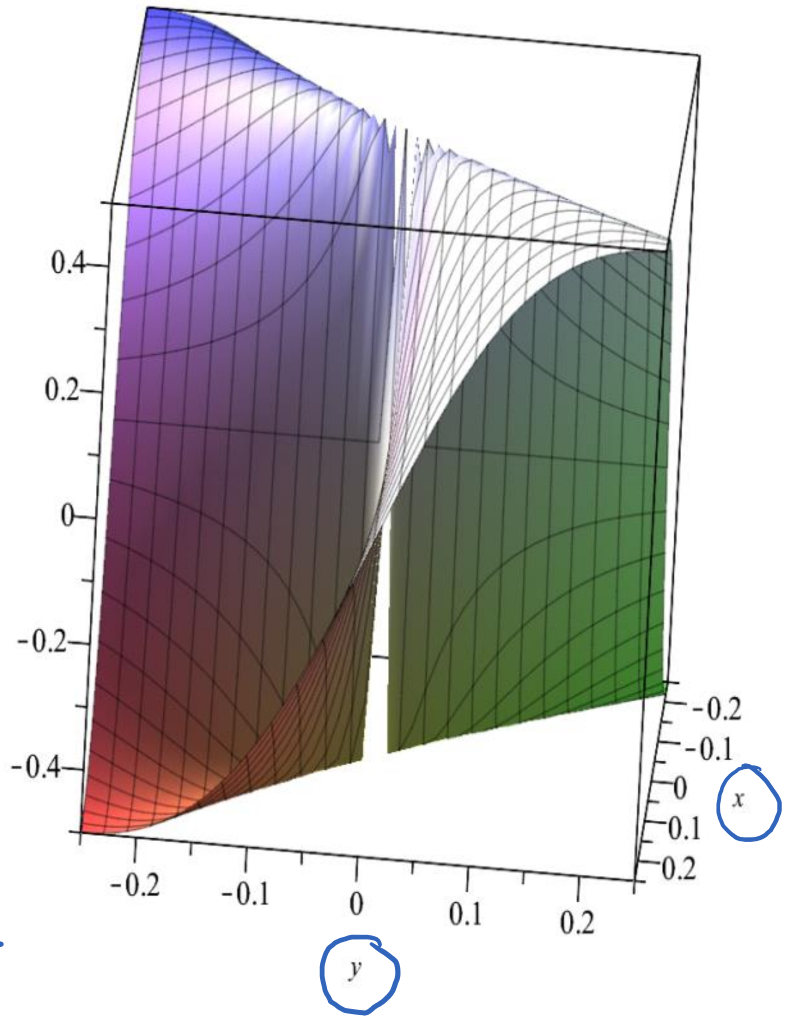
There is no TANGENT PLANE at $(x,y)=(0,0)$

f_x & f_y both exist at (0,0)

$f_x(0,0)=0$

$f_y(0,0)=0$

BUT $f(x,y)$ is NOT continuous at (0,0).



March 24, 2020 7:06 AM

A sufficient condition (but NOT necessary) for the existence of a TANGENT PLANE at a point $(a, b, f(a,b))$:

If f_x and f_y exist at and near (a,b) and both are continuous at and near (a,b), then $f(x,y)$ has a tangent plane at $(a,b,f(a,b))$ given by

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b).$$