§ 14.3 Partial Derivatives.

IMPLICIT DIFFERENTIATION.

$$x^3 + y^3 + z^3 + xyz = 1.$$

implicitly defines z as a function of x and y.

$$z(x,y).$$

Find $\frac{\partial z}{\partial y}$.

Sol'n:

$$0 + 3y^2 + 3z^2 \frac{\partial z}{\partial y} + xz + xy \frac{\partial z}{\partial y} = 0$$

Solve for $\frac{\partial z}{\partial y}$:

$$(3z^2 + xy)\frac{\partial z}{\partial y} = -(3y^2 + xz)$$

$$\frac{\partial z}{\partial y} = -\frac{(3y^2 + xz)}{(3z^2 + xy)}$$

Similarly, find $\frac{\partial z}{\partial x}$.

Show

$$\frac{\partial z}{\partial x} = -\frac{(3x^2 + yz)}{3z^2 + xy}$$

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Function of more than two variables.

$$z = f(x_1, x_2, \ldots, x_n)$$

$$f_{x_i} = \frac{\partial f}{\partial x_i} = \frac{\partial z}{\partial x_i} = D_{x_i} f = D_i f$$

$$= \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_i + h, \ldots, x_n) - f(x_i)}{h}$$

Differentiate wrt x_i, regarding all other variables as constants.

Example. $f(x, y, z) = x^2 e^{y^3 z^4}$

$$\frac{\partial f}{\partial x} = 2x e^{y^3 z^4}$$

$$\frac{\partial f}{\partial y} = x^2 e^{y^3 z^4} (3y^2)$$

$$\frac{\partial f}{\partial z} = x^2 e^{y^3 z^4} (4z^3).$$

Example:

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$\frac{\partial f}{\partial x_j} = 2x_j, \quad j = 1, 2, \ldots, n.$$

e.g.

$$\frac{\partial f}{\partial x_3} = 2x_3.$$

HIGHER ORDER DERIVATIVES.

$f(x,y)$ has 4 kinds of 2nd order partial derivatives.

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \underbrace{\frac{\partial^2 f}{\partial y \partial x}}_{\text{right to left}}$$

left to right.

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

left to right. right to left

Example. $f(x,y) = x^4 + 3x^2y^3 + y^7$

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$f_x = 4x^3 + 6xy^3$; $f_y = 9x^2y^2 + 7y^6$

$f_{xx} = 12x^2 + 6y^3$; $f_{yy} = 18x^2y + 42y^5$

$f_{xy} = \underline{18xy^2}$; $f_{yx} = \underline{18xy^2}$

notice they are equal.

by Clairaut's Th^m.

CLAIRAUT Th^m (1713 – 1765)

Let $V = \{(x,y) : (x-a)^2 + (y-b)^2 < r^2\}$

(V is an open set)

be the open disk of radius r centred at (a, b).

If $f(x,y)$ is a real valued function defined on V such that both f_{xy} and f_{yx} are continuous on V, then $f_{xy} = f_{yx}$ on V.

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NOTE: For "most" functions we see in practice

$$f_{xy} = f_{yx}$$

Sometimes one is easier to evaluate than the other.

e.g. $f(x,y) = e^{x}\cos(x^2) + y$

f_{yx} is easier to evaluate.

3rd order derivatives.

$f(x,y)$ has 8, 3rd order derivatives

$$f_{xxx} \quad \underbrace{f_{xxy} \quad f_{xyx} \quad f_{yxx}}_{\text{same by Clairaut (if th}^{m}\text{ satisfied.}}$$

$$f_{yyy} \quad \underbrace{f_{yxy} \quad f_{yyx} \quad f_{xyy}}$$

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Example. $f(x,y) = x^2y^3$

$f_x = 2xy^3$ $\qquad$ $f_y = 3x^2y^2$

$f_{xx} = 2y^3$ $\qquad$ $f_{yy} = 6x^2y$

$f_{xy} = 6xy^2$ $\qquad$ $f_{yx} = 6xy^2$

$f_{xxx} = 0$ $\qquad$ $f_{yyy} = 6x^2$

$f_{xxy} = 6y^2 = f_{xyx} = f_{yxx}$

$f_{yyx} = 12xy = f_{yxy} = f_{xyy}$

A PARTIAL DIFFERENTIAL EQUATION (PDE) is an equation involving a function of several variables and its partial derivatives.

Laplace eq'n: $f_{xx} + f_{yy} = 0$
(2nd order PDE)

Sol'n. of a PDE is all functions that satisfy the PDE.

Some solutions:

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(i) $f(x,y) = x^2 - y^2$

$f_x = 2x \qquad f_y = 2y$

$f_{xx} = 2 \qquad f_{yy} = 2$

$f_{xx} + f_{yy} = 2 - 2 = 0.$ ✓

(ii) $f(x,y) = x^3 - 3xy^2$

$f_x = 3x^2 - 3y^2 \qquad f_y = -6xy$

$f_{xx} = 6x \qquad f_{yy} = -6x$

$f_{xx} + f_{yy} = 6x - 6x = 0$ ✓

(iii) $f(x,y) = e^x \cos(y)$

$f_x = e^x \cos(y) \qquad f_y = -e^x \sin(y)$

$f_{xx} = e^x \cos(y) \qquad f_{yy} = -e^x \cos(y)$

$f_{xx} + f_{yy} = 0$ ✓

WAVE EQUATION $f(x,t)$

x spatial variable

t time.

$\frac{1}{c^2} f_{tt} = f_{xx}$

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Sol'n. $f(x,y) = \sin(x+ct)$

check.

HEAT EQUATION. $f(x,t)$

$f_t = k f_{xx}$ $k>0$ constant.