March 17, 2020 10:00 AM

§14.2 Limits & Continuity.

Part 1. Limits & Squeeze Th^m.

Def'n Let $f(x,y)$ be a function with domain $D \subseteq \mathbb{R}^2$.

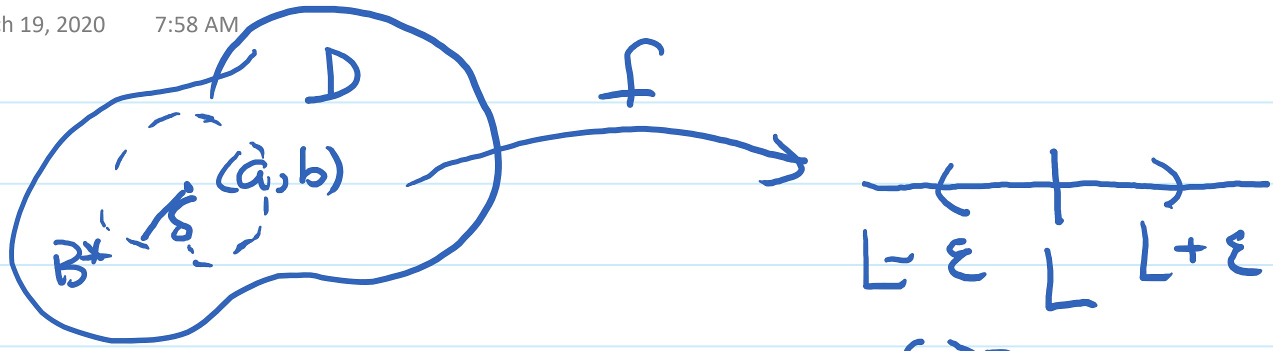
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if the limit exists and is finite.

"the limit of $f(x,y)$ as (x,y) approaches (a,b) is equal to L".

NOTE: We do not require (a,b) to be in D.

March 19, 2020 7:58 AM



$\varepsilon > 0$.

$$B^* = \{(x,y) : (0 < ((x-a)^2+(y-b)^2)^{1/2} < \delta\}.$$

↑ strict

so (a,b) does not need to be in B^*.

<u>BEWARE</u> In order for the limit to exist the result L must be independent of how (x,y) approaches (a,b).

Example. Show $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ DNE

(DNE does not exist).

Sol'n.

March 19, 2020 8:09 AM

Let $(x,y) \to (0,0)$ along the x-axis

i.e. $y=0$, $x \to 0$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-0^2}{x^2+0^2} = 1.$$

Let $(x,y) \to (0,0)$ along the y-axis

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{0^2-y^2}{0^2+y^2} = -1$$

Since $1 \neq -1$, the limit DNE.

Example: Show $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ DNE.

Sol'n. First we approach $(0,0)$ along lines through $(0,0)$ i.e. $y=mx$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy^2}{x^2+y^4} = \lim_{(x,mx)\to(0,0)} \frac{x(mx)^2}{x^2+(mx)^4}$$

$$= \lim_{x\to 0} \frac{x(m^2x^2)}{x^2+m^4x^4}$$

$$= \lim_{x\to 0} \frac{x\cancel{x^2}\,m^2}{\cancel{x^2}(1+m^4x^2)} = 0$$

Now approach $(0,0)$ along a parabola, $x=y^2$:

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^2y^2}{(y^2)^2+y^4} = \frac{1}{2}$$

$$0 \neq \frac{1}{2}$$

$\therefore$ the limit DNE.

§2.3 SQUEEZE Th^m. for functions of 1 variable.

SQUEEZE Th^m for 2 functions of 2 variables.

March 19, 2020 8:21 AM

Let $h(x,y)$, $f(x,y)$ and $g(x,y)$ be functions such that

$$h(x,y) \le f(x,y) \le g(x,y)$$

and

$$\lim_{(x,y)\to(a,b)} h(x,y) = L$$

&

$$\lim_{(x,y)\to(a,b)} g(x,y) = L.$$

Then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Show $\displaystyle\lim_{(x,y)\to(0,0)} \underbrace{\frac{3x^2y}{x^2+y^2}}_{= f(x,y)} = 0.$

$$f(x,y) = \frac{3x^2y}{x^2+y^2} \le 3|y| \frac{x^2}{\underbrace{x^2+y^2}} \le \underbrace{3|y|}_{= g(x,y)}$$

$$\le 1$$

$\therefore \quad f(x,y) \leq g(x,y)$

and

$$\lim_{y\to 0} \underset{\overset{\parallel}{g(x,y)}}{3|y|} = 0.$$

$$f(x,y) = \frac{3x^2y}{x^2+y^2} \geq -3|y|\frac{x^2}{x^2+y^2} \geq \underset{\overset{\parallel}{h(x,y)}}{-3|y|}$$

$$f(x,y) \geq h(x,y)$$

$$\lim_{y\to 0} h(x,y) = \lim_{y\to 0} -3|y| = 0.$$

$$\therefore \quad h(x,y) \leq f(x,y) \leq g(x,y)$$

$h(x,y) \to 0$ as $(x,y)\to(0,0)$ $\qquad$ $g(x,y) \to 0$ as $(x,y)\to(0,0)$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = (0,0).$$

By the SQUEEZE Th^{m}.

March 18, 2020 9:35 PM

________________Von Neumann

The following problem can be solved either the easy way or the hard way.

Two trains 200 miles apart are moving toward each other; each one is going at a speed of 50 miles per hour. A fly starting on the front of one of them flies back and forth between them at a rate of 75 miles per hour. It does this until the trains collide and crush the fly to death. What is the total distance the fly has flown?

The fly actually hits each train an infinite number of times before it gets crushed, and one could solve the problem the hard way with pencil and paper by summing an infinite series of distances. The easy way is as follows: Since the trains are 200 miles apart and each train is going 50 miles an hour, it takes 2 hours for the trains to collide. Therefore the fly was flying for two hours. Since the fly was flying at a rate of 75 miles per hour, the fly must have flown 150 miles. That's all there is to it.

When this problem was posed to John von Neumann, he immediately replied, "150 miles."

"It is very strange," said the poser, "but nearly everyone tries to sum the infinite series."

"What do you mean, strange?" asked Von Neumann. "That's how I did it!"

______________Von Neumann