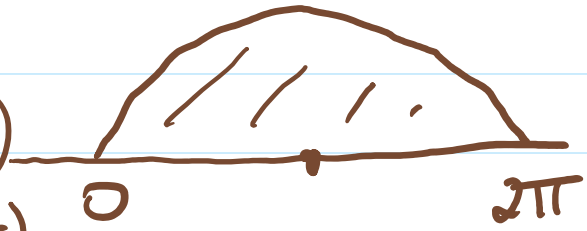
§10.2 cont'd.

Area under a parametric curve.

Example. Find the AREA under 1 arch of the CYCLOID.



$$x = r(\theta - \sin\theta) = f(\theta)$$
$$y = r(1 - \cos\theta) = g(\theta)$$
$$0 \le \theta \le 2\pi$$

Aside: Recall from last lecture.

$$y = F(x)$$
$$a \le x \le b.$$

$$A = \int_a^b F(x)\,dx \qquad x = f(t), \quad y = g(t)$$

$$y = F(x) = g(t)$$
$$dx = f'(t)\,dt$$

$$\int_{t=f(a)}^{f(b)} g(t)\,f'(t)\,dt$$

A of 1 arch of the cycloid

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$$A = \int_{\theta=0}^{2\pi} g(\theta)\, f'(\theta)\, d\theta$$

$$= \int_0^{2\pi} r(1-\cos(\theta))\, r(1-\cos\theta)\, d\theta$$

$$f'(\theta)\, d\theta = r(1-\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} r^2 (1-\cos\theta)^2\, d\theta$$

$$= \int_0^{2\pi} r^2 (1 - 2\cos\theta + \cos^2\theta)\, d\theta.$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= r^2 \left(\theta - 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right)\Bigg|_0^{2\pi}$$

$$= r^2\left(\left(2\pi - 0 + \frac{1}{2}(2\pi + 0)\right) - 0\right)$$

$$= \underline{3r^2\pi}$$

(3 times the area of the circle used to create the cycloid.

Area of an ELLIPSE.

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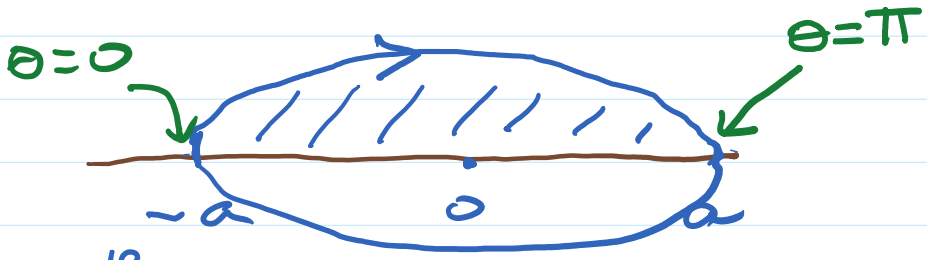
$$x = -a\cos\theta = f(\theta)$$
$$y = b\sin\theta = g(\theta)$$
$$0 \le \theta \le \pi$$



"−" before the a in x= is so the parametric curve goes from left to right.

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = (-\cos\theta)^2 + (\sin\theta)^2 = 1.$$

$$dx = f'(\theta)\,d\theta = a\sin\theta\,d\theta$$

$$\text{Area}\left(= 2\int_{x=-a}^{a} y\,dx\right) = \int_{\theta=0}^{\pi} \underbrace{b\sin\theta}_{g(\theta)}\ \underbrace{a\sin\theta\,d\theta}_{f'(\theta)d\theta}.$$

$$= 2ab\int_0^{\pi} \sin^2\theta\,d\theta$$

$$= \cancel{2}ab\int_0^{\pi} \cancel{\tfrac{1}{2}}\left(1-\cos^2(2\theta)\right)d\theta$$

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$$= (ab)\left(\theta - \frac{\sin(2\theta)}{2}\right)\Big|_0^{\pi}$$

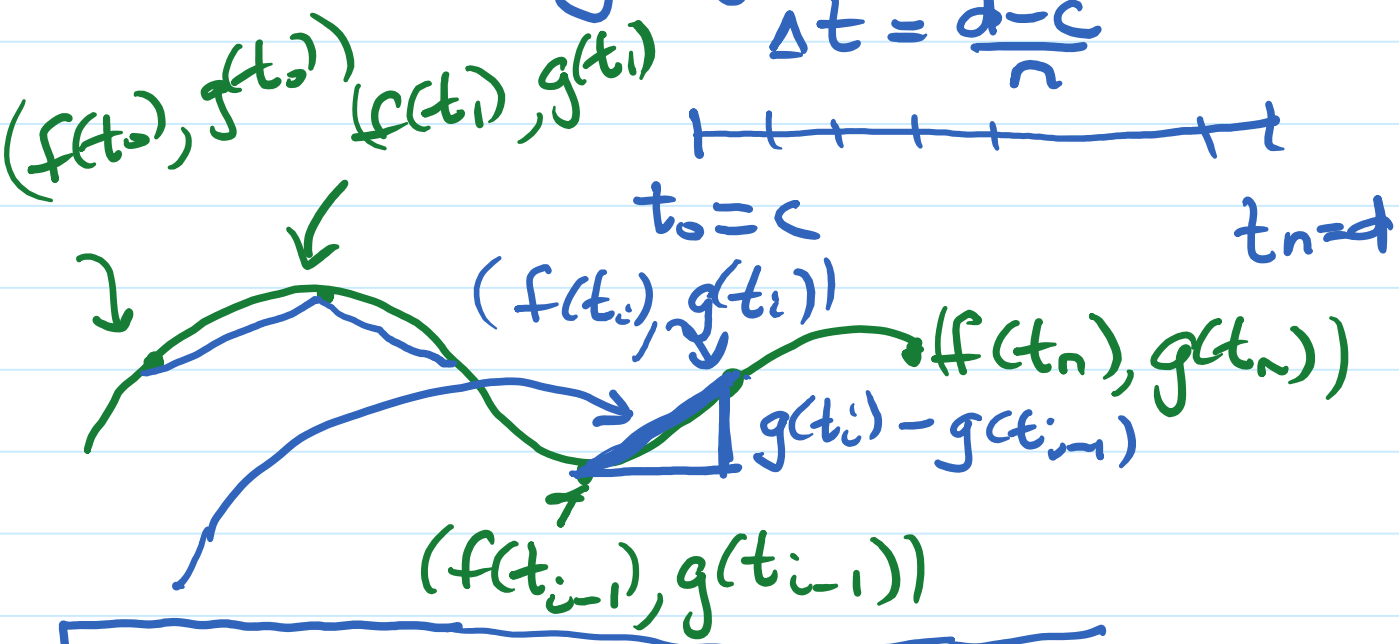
$$= (ab)\left((\pi - 0) - (0)\right)$$

$$= \underline{ab\pi}$$ area of the ellipse.

Arc Length of a Parametric Curve.

$x = f(t)$, $y = g(t)$ $\quad c \le t \le d$.

$\Delta t = \frac{d-c}{n}$



$$\sqrt{(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2}$$

Approx length of the curve from $t = c = x_0$ to $t = d = t_n$.

= sum of the length of the segments

$$\approx \sum_{i=1}^{n} \left((f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2 \right)^{\frac{1}{2}}$$

Mean Value Th^m. and t_i^{**}

there exists t_i^{*}, so that

$$\frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}} = f'(t_i^{*})$$

$$\frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}} = g'(t_i^{**})$$

Approx length.

$$\approx \sum_{i=1}^{n} \left(f'(t_i^{*})^2 + g'(t_i^{**})^2 \right)^{1/2} \Delta t$$

Exact length.

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(f'(t_i^{*})^2 + g'(t_i^{**})^2 \right)^{1/2} \Delta t$$

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This is a Reimann sum

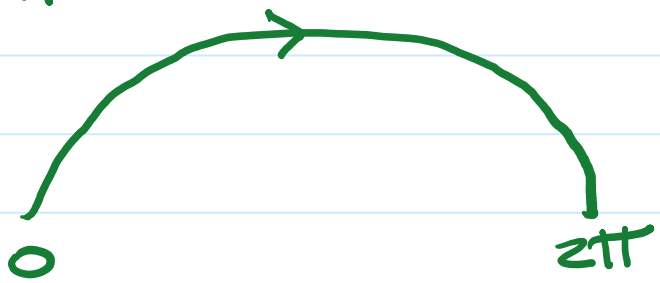
$$\text{Arc length} = \int_c^d \left(f'(t)^2 + (g'(t))^2\right)^{\frac{1}{2}} dt .$$

$$\begin{cases} dx = f'(t)\,dt \\ dy = g'(t)\,dt \\ (dx)^2 + (dy)^2 = (f'(t))^2 + (g'(t))^2 (dt)^2 \end{cases}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt .$$

Arc length of 1 arch of a CYCLOID.

$$x = r(\theta - \sin\theta) = f(\theta)$$
$$y = r(1 - \cos\theta) = g(\theta)$$
$$0 \le \theta \le 2\pi .$$



$$(f'(\theta)^2 + g'(\theta)^2)$$

$$= ((r(1-\cos\theta))^2 + (r\sin\theta)^2$$

$$= r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)$$

$$= r^2(2 - 2\cos\theta)$$

$$= 2r^2(1-\cos\theta)$$

$$= 2r^2\left(2\sin^2\left(\frac{\theta}{2}\right)\right) = 4r^2\sin^2\left(\frac{\theta}{2}\right)$$

using $1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$

$$= \left(2r\sin\left(\frac{\theta}{2}\right)\right)^2$$

Arc length

$$= \int_0^{2\pi} \left((f'(\theta))^2 + (g'(\theta))^2\right)^{1/2} d\theta$$

$$= \int_0^{2\pi} 2r\sin\left(\frac{\theta}{2}\right) d\theta.$$

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$$= 2r\left(-2\cos\left(\frac{\theta}{2}\right)\right)\Big|_0^{2\pi}$$

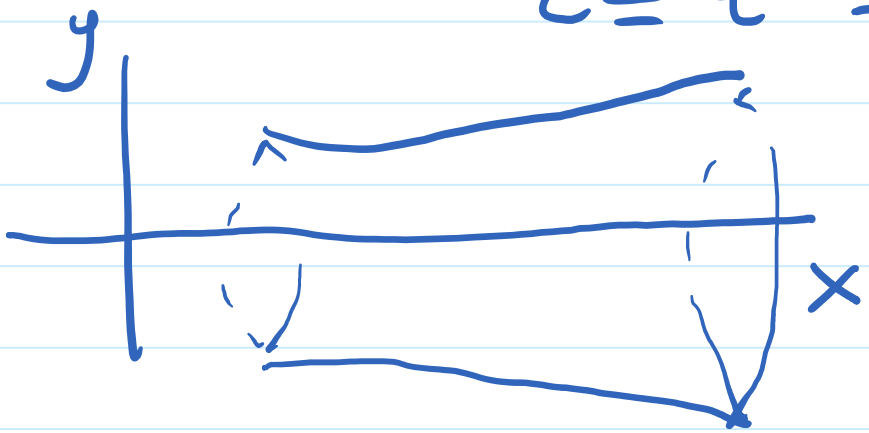
$$= 2r\left((-2)\left((-1)-(1)\right)\right).$$

$$= 2r\left((-2)(-2)\right)$$

$$= \underline{8r}.$$

Surface AREA of REVOLUTION.

$x = f(t)$, $y = g(t)$.
$c \le t \le d$.



Assume curve ① is traversed only once from c to d.

② $g(t) \geq 0$
$c \le t \le d$.

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Surface AREA.

$$= \int 2\pi y \, ds \qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

arc length

$$= \int_{t=c}^{d} 2\pi \, g(t) \underbrace{\left((f'(t))^2 + (g'(t))^2 \right)^{1/2} dt}_{ds}.$$

Example.

$x = r\cos\theta, \qquad 0 \le \theta \le \pi.$

$y = r\sin\theta.$