M1ZB3 Lecture 24 (C02) Dr. Wolkowicz March 6

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# §10.2 Calculus of Parametric Curves (cont'd)

$$x = f(t), \quad y = g(t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \left( \frac{g'(t)}{f'(t)} \right).$$

Example. $x = \cos t$, $y = \sin t$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{\cos t}{-\sin t} = -\cot t$$

NOTE: $\frac{dy}{dx}$ is the SLOPE of the tangent line to the curve.

If $t = \pi k$, $k$ integer.

$$\left.\frac{dx}{dt}\right|_{t=\pi} = -\sin(\pi k) = 0$$

$\therefore$ vertical slope.

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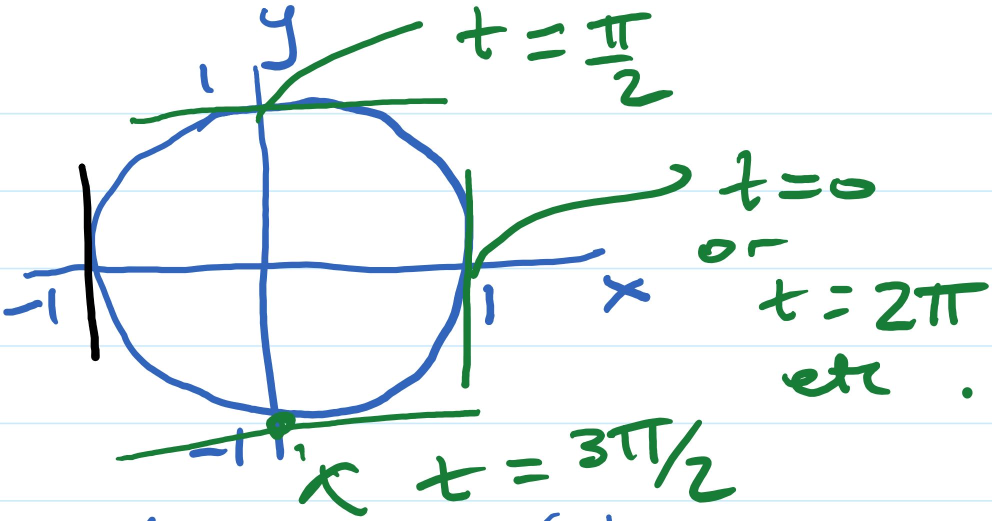

$$\frac{dy}{dt} = \cos t \qquad t = \frac{(2k+1)\pi}{2}$$

k integer

$$\left.\frac{dy}{dt}\right|_{t=\frac{(2k+1)\pi}{2}} = \cos\left(\frac{(2k+1)\pi}{2}\right. = 0.$$

∴ horizontal slope.

Example. $\underbrace{x=t^2}_{x\geq 0}$, $y = t^3 - 3t$
$= t(t^2-3)$

I. Can eliminate $t$
to find $y = F(x)$

$$y^2 = t^2(t^2-3)^2 = x(x-3)^2$$

OR.

$x = t^2$ $y = t(t^2-3)$
$= 0$ $t = \pm\sqrt{3}$.

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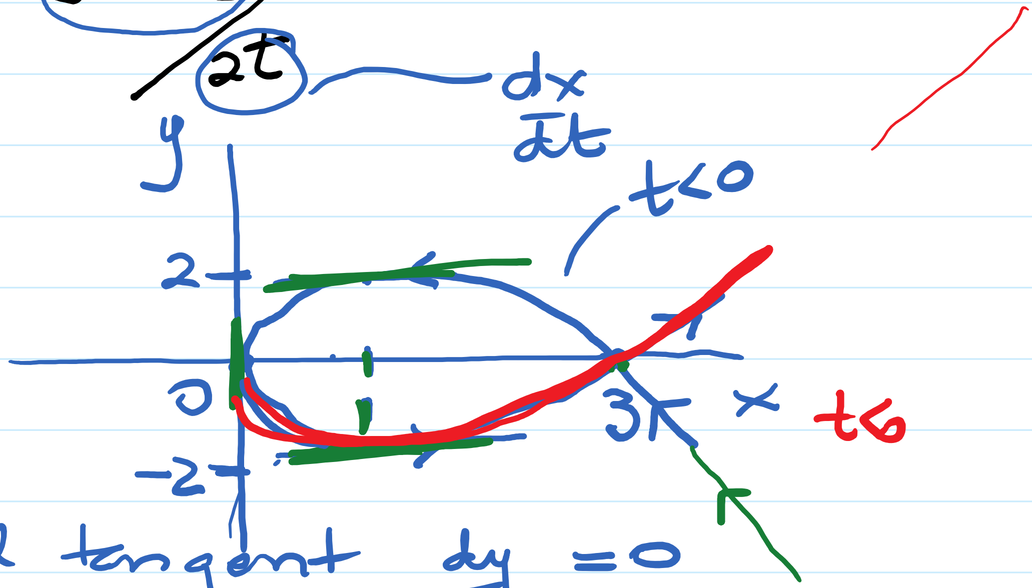
I. $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$.

dy/dt

$\frac{dy}{dx} = \frac{3t^2-3}{2t}$

$\frac{dx}{dt}$

$t = 1$
$\Rightarrow x = 1$



horizontal tangent $\frac{dy}{dt} = 0$

$\frac{dy}{dt} = 3(t^2-1)$ $t = \pm 1$.

$\frac{dx}{dt} = 2t$ $t = 0$ $\left.\frac{dx}{dt}\right|_{t=0} =$

∴ vertical tangent.

$t = \pm\sqrt{3} \Rightarrow x = t^2 = 3$.

$t = \sqrt{3}$ slope is

$$\left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \left.\frac{3t^2-3}{2t}\right|_{t=\sqrt{3}} = \frac{3(3)-3}{2\sqrt{3}}$$

$$= \sqrt{3} > 0.$$

$t = -\sqrt{3}$ $$\left.\frac{dy}{x}\right|_{t=-\sqrt{3}} = -\sqrt{3} < 0$$

## Second Derivatives:

$x = f(t)$ $y = g(t)$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \quad *$$

$$\ast\ \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dt} \Big/ \frac{dx}{dt}\right)}{\frac{dx}{dt}}\ \ast$$

Quotient Rule.

$$= \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}$$

Example. $x = t^2$, $y = t^3 - 3t$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dt} \Big/ \frac{dx}{dt}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t}$$

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$$= \frac{(6t)(2t) - 2(3t^2-3)}{(2t)^3}$$

$$= \frac{6t^2+6}{8t^3}$$

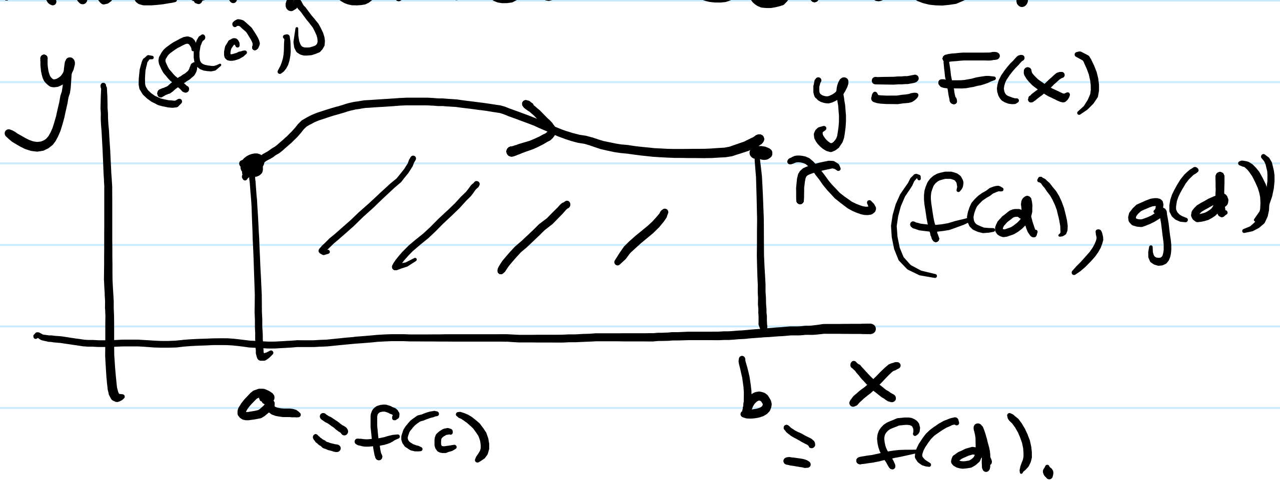
$>0$ if $t>0$

concave up

$<0$ if $t<0$

concave down.

AREA under Curve.



$x = f(t)$, $y = g(t)$

Assume $f$ is increasing in $t$

we traverse the curve from left to right.

$c \leq t \leq d.$

$f(c) = a \qquad f(d) = b.$

$$A = \int_{x=a}^{b} F(x)\,dx = \int_{t=c}^{d} g(t) f'(t)\,dt$$

$y = F(x) = g(t)$

$x = f(t)$

$dx = f'(t)\,dt$

$$\boxed{A = \int_{c}^{d} g(t) f'(t)\,dt}$$