M1ZB3 Lecture 22 (C02) Dr. Wolkowicz March 3

March 3, 2020 11:30 AM

§9.5 Linear Equations and §3.8 Application: Newton's Law of Cooling.

Example: A hot coffee initially at 95°C cools to 80°C in 5 minutes while sitting at room temperature of 21°C. When will the coffee be 50°C?

Sol'n: Let $T(t)$ denote the temperature of the object in °C at time t minutes.

Let $T_s = 21°C$ denote the temperature of the room.

$$\frac{dT}{dt} = k(T - T_s)$$

March 3, 2020 11:20 AM

IVP $\begin{cases} \dfrac{dT}{dt} = k(T-21) \\ T(0) = 95°C, \end{cases}$

$T(5) = 80°C.$

separable (see §3.8).
1^{st} order linear.

(1) $\dfrac{dT}{dt} - kT = -21k$

1^{st} order linear in
STANDARD FORM.

(2) Integrating factor

$P(t) = -k$

$I(t) = e^{\int^t -k\,ds} = e^{-kt}$

(3) key $\left(e^{-kt}T\right)' = (-21k)e^{-kt}$

(4) Integrate: $e^{-kt}T = \int^t -21k\,e^{-ks}\,ds$

$$e^{-kt}T = \frac{-21\cancel{k}}{-\cancel{k}}e^{-kt} + C$$

$$e^{-kT}T = 21e^{-kt} + C$$

$$T = 21 + Ce^{kt} \quad C \text{ arb.}$$

General Sol'n.

Find C using the I.C: $T(0)=95$.

$$95 = T(0) = 21 + C\cancel{e^{0t}}$$

$$C = 74.$$

Sol'n of IVP:

$$T(t) = 21 + 74e^{kt}$$

Find k (to find when $T(t) = 50°$) given

$$T(5) = 80.$$

$$80 = T(5) = 21 + 74e^{5k}$$

$$59 = 74e^{5k}$$

March 3, 2020 11:44 AM

$$e^{5k} = 59/74$$

$$5k = \ln(59/74)$$

$$k = \frac{1}{5}\ln(59/74).$$

$$T(t) = 21 + 74e^{\frac{1}{5}\ln\left(\frac{59}{74}\right)t}$$

When is the coffee at 50°C.

$$50 = 21 + 74e^{\frac{1}{5}\ln\left(\frac{59}{74}\right)\bar{t}}$$

$$\frac{29}{74} = e^{\frac{1}{5}\ln(59/74)\bar{t}}$$

$$\ln\left(\frac{29}{74}\right) = \frac{1}{5}\ln\left(\frac{59}{74}\right)\bar{t}$$

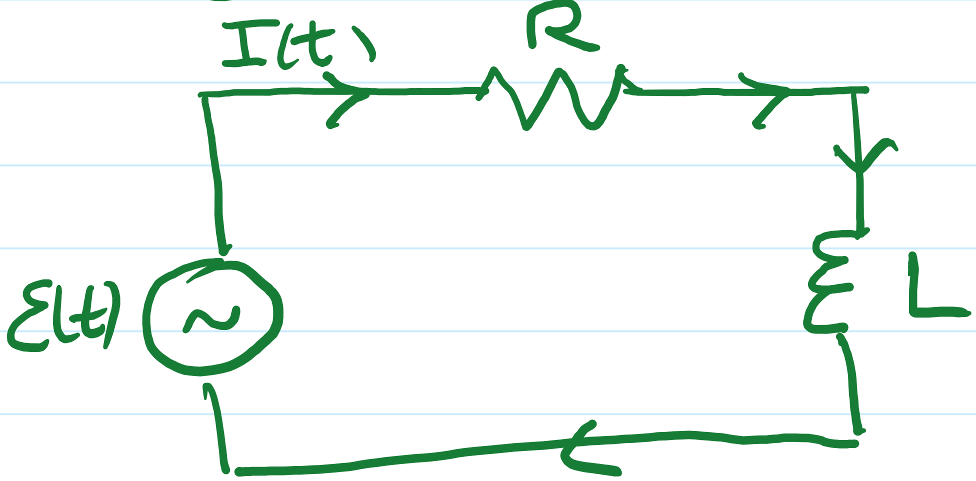
$$\bar{t} = 5\ln\left(\frac{29}{74}\right) \Big/ \ln\left(\frac{59}{74}\right)$$

$$\approx 20.68 \text{ minutes}$$

Exampe.

March 3, 2020 11:49 AM

Electric Circuit.



R = resistance in ohms
L = inductance in henrys.
I(t) = current in amps

$\mathcal{E}(t)$ = electromotive force in volts.

Kirchoff's Voltage Law:
The directed sum o- electrical potential differences around any closed circuit is zero.

$$RI + L\frac{dI}{dt} = \mathcal{E}(t).$$

March 3, 2020 11:54 AM

linear 1^{st} order ODE.

Sol'n:

(1) Standard form.

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{\mathcal{E}(t)}{L}$$

If R, L are constants.

(2) Integrating factor.

$$e^{\int \frac{R}{L} ds} = e^{\frac{R}{L}t}$$

(3)
$$\left(e^{\frac{R}{L}t} I\right)' = e^{\frac{R}{L}t} \frac{\mathcal{E}(t)}{L}$$

(4)
$$e^{\frac{R}{L}t} I = \int^{t} e^{\frac{R}{L}s} \frac{\mathcal{E}(s)}{L} ds + C$$

(5)
$$I(t) = \frac{e^{-\frac{R}{L}t}}{L} \int^{t} e^{\frac{R}{L}s} \mathcal{E}(s) ds + \underbrace{e^{-\frac{R}{L}t} C}$$

March 3, 2020 12:00 PM

The $Ce^{-\frac{R}{L}t}$ is called the <u>TRANSIENT</u>

since $Ce^{-R/L\,t} \to 0$ as $t \to \infty$.

Example.

IVP $\begin{cases} \cos\theta \dfrac{dr}{d\theta} + r\sin\theta = 1, \\ r(3\pi) = 2. \end{cases}$

(1) STANDARD FORM.

$$\frac{dr}{d\theta} + \tan\theta\, r = \sec\theta.$$

($\cos\theta \neq 0$

i.e. $\theta = \frac{(2k+1)\pi}{2}$

$k \in \mathbb{Z}$.

(2) Integrating factor

$$I(\theta) = e^{\int \tan\theta\, d\theta}$$

$$= e^{-\ln|\cos\theta|}$$

$$= \frac{1}{|\cos\theta|} = |\sec\theta|.$$

March 3, 2020 12:06 PM

Take $I(t) = \sec(\theta)$.

(3). $(\sec\theta\, r)' = \sec^2\theta$

(4) $\sec\theta\, r = \int^{\theta} \sec^2(s)\,ds$

$\sec(\theta)\, r = \tan\theta + c$

$r = \cos\theta\,(\tan\theta + c)$

$\theta = \frac{(2k+1)\pi}{2}, \; k \in \mathbb{Z}$.

IVP. $r(3\pi) = 2$.

$2 = r(3\pi) = \cos(3\pi)(\tan(3\pi) + c$

$2 = (-1)(0 + c)$

$\Rightarrow \quad c = -2$.

$\therefore \; r(t) = \cos\theta(\tan\theta - 2)$

$\frac{5\pi}{2} < \theta < \frac{7\pi}{2}$

Example:

March 3, 2020 12:11 PM

$$(6uv^2)\,dv + 2(v^3+1)\,du = 0.$$

linear in dep. var u.

i.e. $u(v)$

$$\frac{du}{dv} + u\left(\frac{6v^2}{2(v^3+1)}\right) = 0$$

<u>IS</u> 1st order linear

NOT linear in dep. var v.

i.e. $v(u)$

$$\frac{dv}{du} + \frac{2(v^3+1)}{6uv^2} = 0$$

$$\frac{dv}{du} + \underbrace{\frac{1}{3u}}_{P(u)} v = \underbrace{-\frac{1}{3uv^2}}_{\text{This is a function of } u \text{ and } v}$$

$Q(u,v) \neq Q(u)$ only

but would have to be a function of only the dependent variable u.

i.e. $v' + P(u)v = Q(u)$