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§ 3.8.

Michael Chrichton (1969)
The Andromeda Strain

"The mathematics of uncontrolled growth are frightening. A single cell of E. coli would under ideal circumstance, divide every **20 minutes**. That is not particularly disturbing until you think about it, but the fact is that bacteria multiply geometrically: one becomes 2, 2 becomes 4, 4 becomes 8, and so on. In this way, it can be shown that in a single day, one cell of E. coli could produce a super-colony in size and weight equal to the **entire planet earth**."

Actually, it would take approximately **2 days**.

Average mass of 1 E coli cell is approx. 10^{-12} gm

Mass of the earth is approx. (5.976) 10^{24} kg

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Method I : As a difference equation.

Let x_n be the # of E coli.

3, 20 minute periods in an hour.

$\frac{1}{3}$ hr = 20 min.

$x_{n+1} = 2x_n$

$x_0 = 1$

n = 1 20 min.

n = 2 40 min.

n = 3 60 min 1 hr.

Soln

$x_n = 2^n x_0$

$2^n \underbrace{10^{-12}\ gm}_{1\ Ecli.} = 5973.6 \times 10^{24}\ gm.$

$2^n = 5973.6 \times 10^{36}$

$n \ln 2 = \ln(5973.6 \times 10^{36})$

$n = \frac{\ln(5973.6 \times 10^{36})}{\ln 2}.$

n 20 min periods

hrs $\frac{n}{3} = \frac{\ln(5973.6 \times 10^{36})}{3 \ln 2}$ hrs

≈ 41 hrs $\simeq$ 1 day 17 hrs.

Method II. as ODE.

Sol'n: $N' = kN$

$N(0) = 10^{-12}$ gm

$N(t)$ mass of E coli at time t in (hr).

Doubling time is 20 min $= \frac{1}{3}$ hr.

$N(\frac{1}{3}) = 2N(0)$

Find k.

$N(\frac{1}{3}) = 2N(0)$

$\|$

$N'(\frac{1}{3}) = kN(\frac{1}{3})$

$N(t) = N(0)e^{kt}$

$N(\frac{1}{3}) = N(0)e^{k/3} = 2N(0)$

$\frac{k}{3} = \ln 2 \Rightarrow k = 3\ln 2$

$$N(t) = N(0)\, e^{(3\ln 2)t}$$

t

10^{-12}

Find $\bar{t}$

$$N(\bar{t}) = 10^{-12}\, e^{(3\ln 2)\bar{t}} = 5973.6 \times 10^{24}\ \text{gm}$$

$$e^{(3\ln 2)\bar{t}} = 5973.6 \times 10^{36}$$

$$(3\ln 2)\bar{t} = \ln(5973.6 \times 10^{36})$$

$$\bar{t} = \frac{\ln(5973.6 \times 10^{36})}{3\ln 2}$$

Same as using Method I.

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§ 9.5. Linear ODE.

A first ORDER linear ODEs is of the form.

$$* \quad \frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$, $Q(x)$ are given functions.

$*$ is in STANDARD FORM.

RECALL. The Fundamental Thm of Calculus.

$$\frac{d}{dx}\int_a^{h(x)} f(s)\,ds = f(h(x))\,h'(x)$$

$$\frac{d}{dx}\, e^{\int^x P(s)\,ds} = e^{\int^x P(s)\,ds} P(x)$$

To Solve

$$A(x)\,y'(x) + B(x)\,y(x) = C(x)$$

STANDARD FORM:

$$y'(x) + \frac{B(x)}{A(x)}\,y(x) = \frac{C(x)}{A(x)}$$

$$\frac{B(x)}{A(x)} = P(x) \qquad \frac{C(x)}{A(x)} = Q(x)$$

(1) Put ODE in STANDARD FORM

$$y' + P(x)y = Q(x)$$

(2) Find an "Integrating Factor"

$$I(x) = e^{\int^x P(s)\,ds}$$

Consider

$$\underbrace{e^{\int^x P(s)ds}\left(y' + P(x)y\right)}_{\left(e^{\int^x P(s)ds}\,y\right)'} = e^{\int^x P(s)ds}\,Q(x)$$

(3) KEY NOTE

$$e^{\int^x P(s)\,ds}\left(y' + P(x)\,y\right) = \left(e^{\int^x P(s)\,ds}\,y\right)'$$

$I(x)$ dep. var.

$$\left(e^{\int^x P(s)\,ds}\,y\right)' = e^{\int^x P(s)\,ds}\,y' + e^{\int^x P(s)\,ds}\,P(x)\,y$$

$$= e^{\int^x P(s)\,ds}\left(y' + P(x)\,y\right)$$

ODE :

$$\left(e^{\int^x P(s)\,ds}\,y\right)' = e^{\int^x P(s)\,ds}\,Q(x)$$

(4) Integrate both sides

$$e^{\int^x P(s)\,ds}\,y = \int^x e^{\int^u P(s)\,ds}\,Q(u)\,du + C$$

Calb.

(5) Solve for y.

$$y(x) = e^{-\int^x P(s)\,ds}$$

$$= e^{-\int^x P(s)\,ds}\left[\int e^{\int^u P(s)\,ds} Q(u)\,du + c\right]$$

$$\therefore y(x) = e^{-\int^x P(s)\,ds}\int^x e^{\int^u P(s)\,ds} Q(u)\,du + \underbrace{e^{-\int^x P(s)\,ds} C}_{C \text{ and } b}$$

GENERAL SOL'N.

BEWARE: C is multiplied by $\frac{1}{I(x)} = e^{-\int^x P(s)\,ds}$.

Example $\frac{dy}{dx} = x + y$ *

IS 1st order linear. (NOTE: NOT Separable)

(1) STANDARD FORM

$$\frac{dy}{dx} - y = x$$

$P(x) = -1$; $Q(x) = x$.

(2) Integrating factor

$$I(x) = e^{\int^x -1\, ds} = e^{-x}$$

(Consider $e^{-x}(y' - y) = e^{-x}x$)

(3) KEY STEP $\left(e^{-x}y\right)' = e^{-x}x$

(4) Integrate

$$e^{-x}y = \int^x e^{-x}x\, dx$$

Integrate by parts.

$$e^{-x}y = -xe^{-x} - e^{-x} + C$$

$$\therefore y(x) = -x\frac{e^{-x}}{e^{-x}} - \frac{e^{-x}}{e^{-x}} + \frac{C}{e^{-x}}$$

$$\boxed{y(x) = -x - 1 + Ce^{x}}$$

C arb.

BEWARE:
C is divided by e^{-x}.

CHECK by subst into the ODE.

§3.8 Newton's Law of Cooling.

Rate of warming or cooling is proportional to the temperature difference between an object and its surroundings.

Let $T(t)$ be the temperature of an object at time t

T_s denote the temperature of the surroundings.

$$\frac{dT}{dt} = k(T - T_s)$$

↑ k constant of proportionality.

NOTE: SEPARABLE & it is 1st order linear.