M1ZB3 Lecture 20 (C02) Dr. Wolkowicz Feb. 27

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Michael Chrichton (1969)
The Andromeda Strain

"The mathematics of uncontrolled growth are frightening. A single cell of E. coli would under ideal circumstance, divide every **20 minutes**. That is not particularly disturbing until you think about it, but the fact is that bacteria multiply geometrically: one becomes 2, 2 becomes 4, 4 becomes 8, and so on. In this way, it can be shown that in a single day, one cell of E. coli could produce a super-colony in size and weight equal to the **entire planet earth**."

Actually, it would take approximately **2 days**.

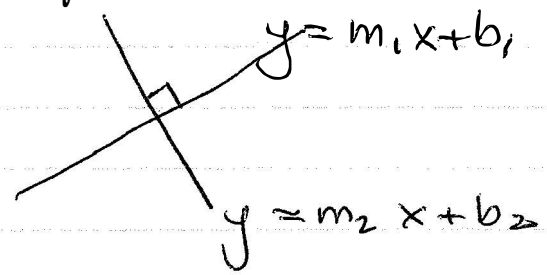
Average mass of 1 E coli cell is approx. 10^{-12} gm

Mass of the earth is approx. $(5.9763)\ 10^{24}$ kg

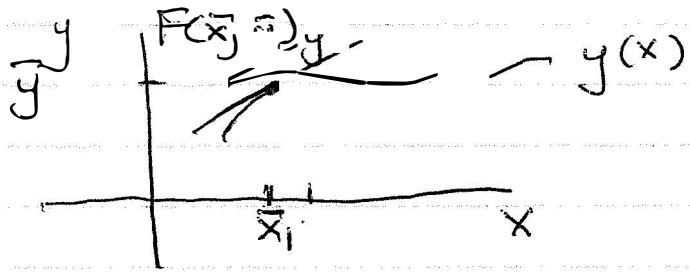
§9.3. Example: Orthogonal Trajectories.

Consider lines. $y = m_1 x + b_1$
$y = m_2 x + b_2$

They are orthogonal (or perpendicular) if $m_2 = -\frac{1}{m_1}$



ODE: $\frac{dy}{dx} = F(x,y)$



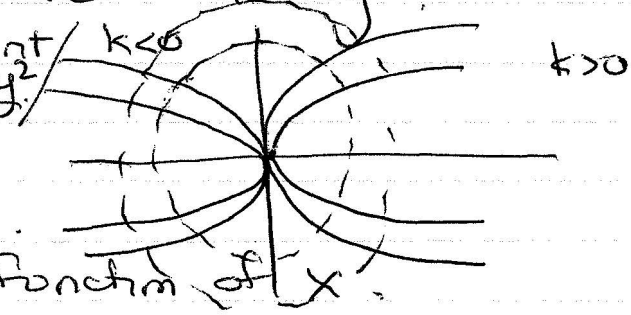
$F(\bar{x},\bar{y})$ is the slope of $y(x)$ at $x = \bar{x}$.

Curves that cross $\perp$ have slope $\frac{-1}{F(x,y)}$.

Aim: Given a family of curves, find the orthogonal trajectorie, i.e. the family of curves that cross these curves orthogonally.

Example. Find the orthogonal trajectorie to the family of curves: $x = ky^2$.

① Find the slope at each point of $x = ky^2$.



Method I: $\frac{x}{y^2} = k$

Use implicit differentiation. thinking of y as a function of x. & eliminate k.

Feb. 27/20

$$\frac{1\,y^2 - 2y\,y'x}{y^{\cancel{4}}} = 0$$

$$\cancel{y}\,(y - 2y'x) = 0 \Rightarrow \boxed{y' = \frac{y}{2x} = F(x,y)}$$

Method II.

Implicit diff.

$x = ky^2 \Rightarrow 1 = k\,2y\,y'$

$\Downarrow \quad \Rightarrow \frac{dy}{dx} = \frac{1}{2ky} = \frac{1}{2\frac{x}{y^2}y} = \frac{y}{2x} = F(x,y)$

$k = x/y^2$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{2x} \leftarrow F(x,y)$$

Family of orthogonal trajectories has slope

$$\frac{dy}{dx} = -\frac{1}{F(x,y)} = -\frac{2x}{y}.$$

$$\frac{dy}{dx} = -\frac{2x}{y}$$

Separable.

$$\int y\,dy = \int -2x\,dx$$

$$\frac{y^2}{2} = -\frac{\cancel{2}x^2}{\cancel{2}} + C, \quad C \text{ arb.}$$

$$x^2 + \frac{y^2}{2} = C, \quad C \text{ arb}$$

Family of ellipses.

gives the orthogonal trajectories,

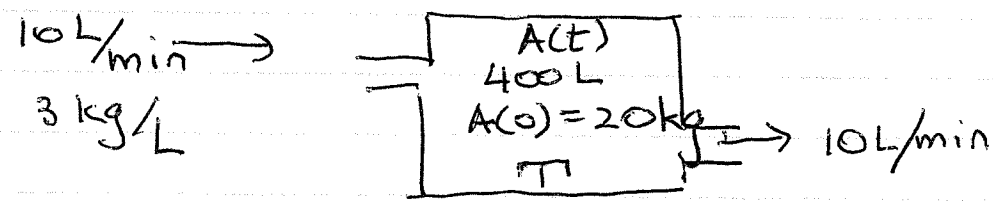
Example: Mixing Problem.

Brine containing 3 kg/L of salt enters a tank filled with 400 L of water containing 20 kg salt.

Brine enters at 10 L/min.
Mixture is well-stirred.
Mixture leaves at the same rate.

Question: How much salt is in the tank after 10 minutes?

Sol'n: Define variables,
Let $A(t)$ denote the amount of salt (in kg) in the tank at time t (in minutes),



$$\underbrace{\frac{dA}{dt}}_{\text{Change in amount salt wrt time}} = \text{rate of amt of salt entering} - \text{rate of amt salt leaving}$$

kg/min kg/min kg/min

$$= \left(\frac{3\text{ kg}}{\cancel{L}}\right)\left(\frac{10\cancel{L}}{\text{min}}\right) - \left(\frac{A(t)}{400\cancel{L}}\right)\frac{10\cancel{L}}{\text{min}}$$

IVP.
$$\begin{cases} \boxed{\dfrac{dA}{dt} = 30 - \dfrac{A(t)}{40}} \\ A(0) = 20 \\ A(10) = ? \end{cases}$$

Separable: $\dfrac{dA}{dt} = \dfrac{1200 - A(t)}{40}$.

$$\int \frac{40\,dA}{1200 - A} = \int dt.$$

Feb. 27/20

$$\Rightarrow -40 \ln|1200-A| = t + C \quad \text{arb}$$

$$\ln|1200-A| = -\frac{1}{40}t \quad -\frac{1}{40}C \text{ "} K \text{ arb.}$$

$$\ln|1200-A| = -\frac{1}{40}t + K$$

$$|1200-A| = e^{-\frac{1}{40}t} e^{K}$$

$$1200 - A = \pm e^{-\frac{1}{40}t} e^{K}$$

$$1200 - A = \tilde{k} e^{-\frac{1}{40}t} \qquad \tilde{k} \neq 0, \text{ arb.}$$

$$A(t) = 1200 - \tilde{k} e^{-\frac{1}{40}t}$$

To find k, use the I.C. $A(0) = 20$.

$$20 = A(0) = 1200 - \tilde{k} e^{-\frac{1}{40}\cdot 0}$$

$$20 = 1200 - \tilde{k}$$

$$\tilde{k} = 1180.$$

$$\therefore A(t) = 1200 - 1180 e^{-t/40} \text{ kg.}$$

ANS After 10 minutes the amount of salt in the tank is

$$A(10) = 1200 - 1180 e^{-10/40} \text{ kg.}$$

$$\approx 281 \text{ kg.}$$