§9.1 Modeling with Differential Equations

Ordinary differential equation (ODE) - equation involving an unknown function and its ~~dervitat~~ derivatives (e.g. 1st, 2nd, 3rd, ...) y', y'', y''' etc.

Def'n. Order of an ODE is the highest derivative in the equation.

Example:

I. $y'(x) = \cos(x^3 + 3y^2(x))$.

1st order.

II. $(y''')^2 e^x + y^4 (y')^5 = \cos(x^6)$.

3rd order

Solution of an ODE
is a function that
satisfies the equation.

Example: Exponential growth ($a>0$)
or decay ($a<0$).

(*) $y' = ay$, "a" a real number.

$(y'(t) = ay(t))$ t time.
y pop. size.

Solution: $y(t) = \underline{Ce^{at}}$, C arb constant

General solution is ALL
possible solutions.

Check it is a solution: LHS = RHS

$$\text{LHS} = y'(t) = aCe^{at}$$
$$\text{RHS} = ay(t) = aCe^{at}$$
✓ LHS = RHS.

Given an
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initial condition (I.C)
(e.g. pop. size at time $t=0$.

IVP $\begin{cases} y' = ay \\ y(0) = 1000 \quad \text{I.C.} \end{cases}$

Initial value problem (IVP).

Expect a unique solution
(or no solution usually.).

The initial condition
determines the
arb. constant.

$y(t) = Ce^{at}$ general sol'n.

$y(0) = 1000$,

$\therefore y(0) = C\cancel{e^{a0}} = 1000$
$\Rightarrow C = 1000.$

$\therefore y(t) = 1000e^{at}$ is the
sol'n of the IVP.

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§9.3. Separable ODEs.

A general first order ODE

$$y' = F(x,y)$$

where $y(x)$.

Example $y' = \underbrace{x^2+y^2}_{F(x,y)}$

NOT SEPARABLE.

Def'n. $y' = F(x,y)$ IS separable if $F(x,y)$ can be written.

$$F(x,y) = g(x)f(y)$$

To solve. $\frac{dy}{dx} = g(x)f(y)$

separate: $\frac{1}{f(y)}\,dy = g(x)\,dx$.

$(f(y) \neq 0)$

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integrate: $\int \frac{1}{f(y)}\, dy = \int g(x)\, dx$

Example. Solve the IVP

$$\begin{cases} \frac{dy}{dx} = x^2/y^4 \\ y(0) = 2. \end{cases}$$

Separable: $\frac{dy}{dx} = (x^2)\left(\frac{1}{y^4}\right)$

$g(x) = x^2$

$f(y) = \frac{1}{y^4}$

Separate:

$$y^4\, dy = x^2\, dx$$

Integrate:

$$\int y^4\, dy = \int x^2\, dx$$

$$y^5/5 = \frac{x^3}{3} + C, \quad C \text{ arb.}$$

(implicit form)

$$y(x) = \left(\frac{5}{3}x^3 + 5C\right)^{1/5}, \quad C \text{ arb.}$$

General sol'n.
(explicit form)

Use the I.C. to find C.

I.C. $y(0) = 2$.

$$\underset{x}{\uparrow} \qquad \underset{y}{\uparrow}$$

$$2 = \left(\frac{5}{3}0^3 + 5C\right)^{1/5}$$

$$2 = (5C)^{1/5}$$

$$32 = 5C \Rightarrow C = \frac{32}{5}.$$

$\therefore$ Sol'n of IVP is.

$$y(x) = \left(\frac{5}{3}x^3 + 32\right)^{1/5}$$

<u>check</u>. (HW).

Example. (Implicit Sol'n).

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$$\frac{dy}{dx} = \frac{e^x}{2y+\cos(y)}. \quad *$$

Separate & integrate:

$$\int (2y + \cos(y))\,dy = \int e^x\,dx$$

$$\cancel{2}\frac{y^2}{\cancel{2}} + \sin y = e^x + C.$$

C arb.

implicit sol'n.
cannot solve for $y(x)$.

CHECK using implicit differentiation: $y(x)$
differentiate wrt x.

$$2y\,y' + \cos y\; y' = e^x + 0$$

$$y'(2y+\cos y) = e^x$$

$$y' = e^x/(2y+\cos y) \quad * \checkmark$$

Example:

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$$\boxed{\frac{dy}{dx} = 3x^4 y}$$

$$\int \frac{1}{y}\, dy = \int 3x^4 \qquad (\underline{y \neq 0})$$

$$\ln|y| = \frac{3}{5}x^5 + C, \quad C \text{ arb.}$$

$$|y| = e^{\frac{3}{5}x^5 + C}$$

$$|y| = e^{\frac{3}{5}x^5} e^{C}$$

$$y = \pm e^{\frac{3}{5}x^5} e^{C}$$

$$\boxed{y = A e^{3/5 x^5}} \qquad A \neq 0, \text{ arb.}$$

$\pm e^{C}$ arb but $\neq 0$.

What if $y = 0$?

If $y(x) = 0$ for all x.

LHS $y' = 0$ RHS $= 0$.

$\therefore$ $y = A e^{3/5 x^5}$, A arb

is general solution.

Example.

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Logistic Growth of a population

$$x' = rx\left(1 - \frac{x}{K}\right).$$

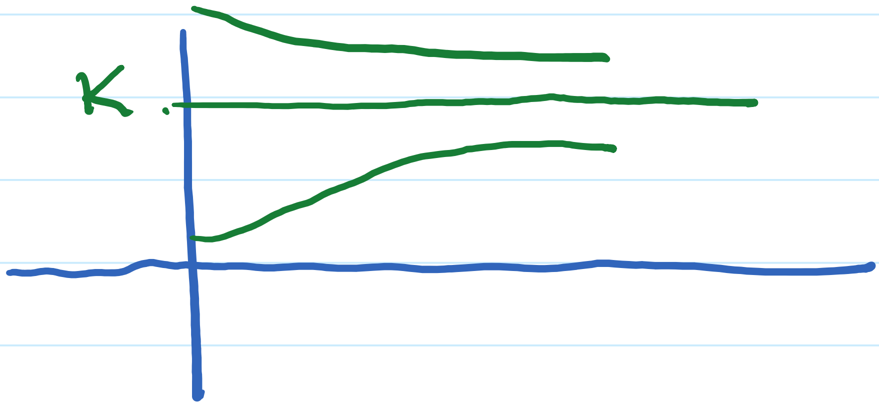
$r>0$, $K>0$

r intrinsic growth rate.

K carrying capacity of environment.

Separable.

$$\int \frac{1}{rx(1-x/K)}\,dx = dt.$$



$x(t) = K$
$x(t) = 0$ } steady state