M1ZB3 Lecture 17 (C02) Dr. Wolkowicz Feb. 13

February 13, 2020

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all } x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for all } x \in \mathbb{R}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}, \quad \text{for all } x \in \mathbb{R}$$

Binomial Series:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad k \in \mathbb{R}$$

with ROC $R = 1$.

where $\displaystyle \binom{k}{n} = \frac{k(k-1)\cdots(k-n+2)(k-n+1)}{n!}$

the Binomial coefficients

Note: If k is an integer $\displaystyle \binom{k}{n} = \frac{k!}{n!(k-n)!}$, and if $n > k$ $\displaystyle \binom{k}{n} = 0$

§11.11 Approximating Functions using Polynomials

Example: $f(x) = \sqrt{x}$, $x \geq 0$.

I Calculate the 2^{nd} order Taylor Polynomial at $a=4$

$$T_2(x) = f(4) + f'(4)(x-4) + f''(4)(x-4)^2/2!$$

$$f(4) = 4^{1/2} = 2$$

$$f'(x) = \frac{1}{2}x^{-1/2}; \quad f'(4) = \frac{1}{2}4^{-1/2} = \frac{1}{4}$$

$$f''(x) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}, \quad f''(4) = -\frac{1}{4}(4)^{-3/2}$$

$$= -\frac{1}{4}\left(\frac{1}{8}\right) = -\frac{1}{32}$$

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{\underbrace{(32)2!}_{64.}}(x-4)^2$$

February 13, 2020 11:40 AM

II Estimate how accurate the approx is to $\sqrt{x}$ for $3 \le x \le 6$.

$$R_n(x) \le |T_n(x) - f(x)| \le \frac{M\,|x-a|^{n+1}}{(n+1)!} \quad |x-a| < d.$$

$$M = \max_{|x-a|<d} |f^{(n+1)}(x)|$$

$$R_2(x) = |f(x) - T_2(x)|$$

$$\le \frac{M}{3!}|x-4|^3 \qquad 3 \le x \le 6$$

$$M = \max_{3 \le x \le 6} |f'''(x)|$$

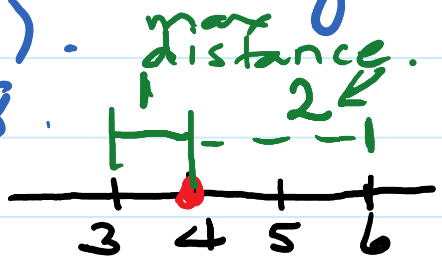
$$f'''(x) = \left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-5/2} = \frac{3}{8}x^{-5/2}$$

$$M = \max_{3 \le x \le 6} \left|\frac{3}{8}x^{-5/2}\right| = \frac{3}{8}(3)^{-5/2}$$

since $\uparrow$ is decreasing on $(3, 6)$.

$$\max_{3 \le x \le 6} |x-4|^3 = 2^3 = 8.$$

$\uparrow$ distance of x from 4.



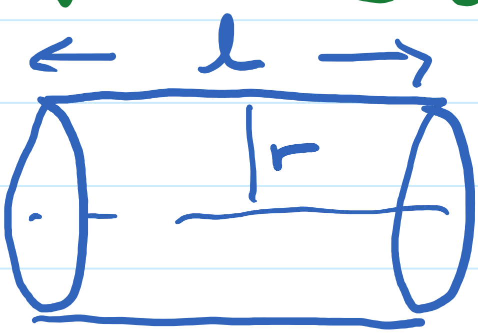
$$R_2(x) \le \frac{\overbrace{\frac{3}{8}(8)^{-5/2}}^{M} \cdot 8}{3!} \qquad 8 \leftarrow |x-a|^3$$

$$= \frac{\sqrt{3}}{54} \approx 0.032075^{+}\ldots$$

$$R_2(x) \le 0.0321$$

§8.2. AREA of SURFACE of REVOLUTION.

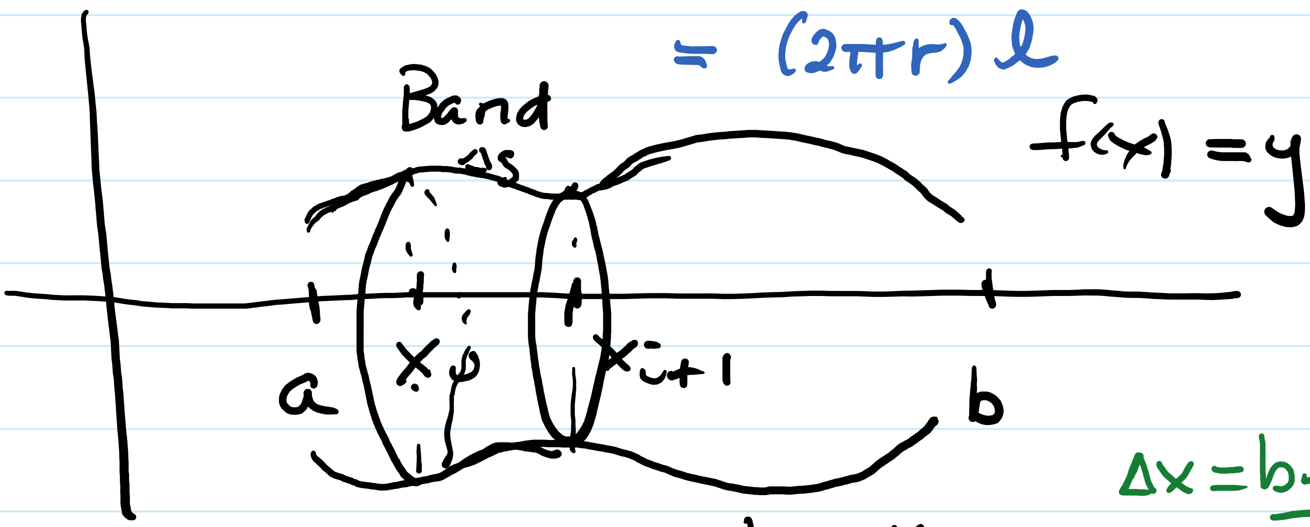
(Recall: §6.2 you found VOLUME of revolution).



Surface AREA.

Circumference $2\pi r$

$= (2\pi r)\ell$



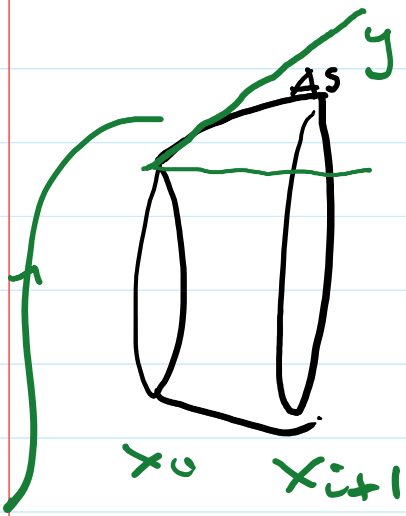
Δs arc length.

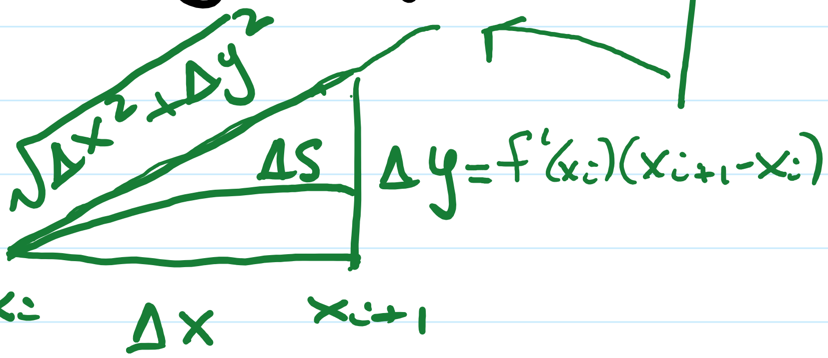
$$\Delta x = \frac{b-a}{n}$$

Revolve $f(x)$ about the x-axis.

$y = f(x_i) + f'(x_i)(x_{i+1} - x_i)$

Arc length (§ 8.1)





Tangent line approximation to the arc length Δs.

$$\Delta s \cong \sqrt{\Delta x^2 + \Delta y^2}$$

$$\Delta y \approx f'(x_i)\,\Delta x.$$

$$= \sqrt{(\Delta x)^2 + (f'(x_i))^2(\Delta x)}$$

$$= \sqrt{1 + (f'(x_i))^2}\;\Delta x$$

Area of SURFACE of REVOLUTION

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{2\pi f(x_i)}_{\text{circumference}} \underbrace{\left(1 + (f'(x_i))^2\right)^{1/2} \Delta x}_{\text{arc length.}}$$

February 13, 2020 12:05 PM

Δs

radius is $f(x)$.

Surface Area

= circumference · arc length

$(2\pi f(x_i))\,\Delta s$

limit of Reimann Sum

$$\lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i)\sqrt{1+(f'(x_i))^2}\,\Delta x$$

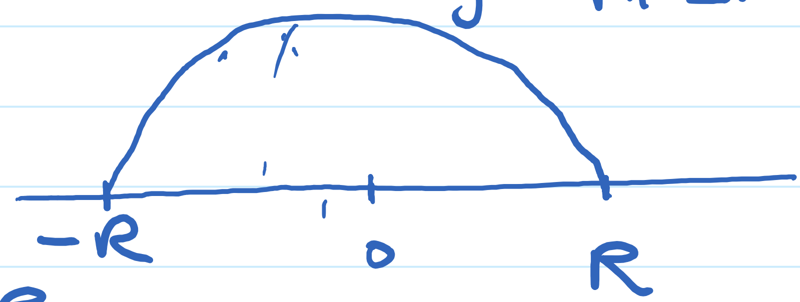
$$\text{Surface AREA} = \int_a^b \underbrace{2\pi f(x)}_{\text{Circumference}} \underbrace{\left(1+(f'(x))^2\right)^{1/2} dx}_{\substack{ds \\ \text{arc length}}}$$

Example. Surface AREA of a SPHERE.

Rotate a Semi-Circle about the x-axis.

February 13, 2020 12:11 PM

$y = f(x)$
$= \sqrt{R^2 - x^2}$



$-R \le x \le R$

$$f'(x) = \frac{1}{\cancel{2}}(R^2 - x^2)^{-1/2}(-\cancel{2}x)$$

$$= \frac{-x}{\sqrt{R^2 - x^2}}$$

$$1 + (f'(x))^2 = 1 + \left(\frac{-x}{\sqrt{R^2 - x^2}}\right)^2$$

$$= 1 + \frac{x^2}{R^2 - x^2}$$

$$= \frac{R^2 - x^2 + x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}.$$

Surface AREA of REVOLUTION:

$$= \int_{-R}^{R} 2\pi f(x)\left(1 + (f'(x))^2\right)^{1/2} dx$$

February 13, 2020 12:14 PM

$$= \int_{-R}^{R} 2\pi \sqrt{R^2 - x^2} \left(\frac{R^2}{R^2 - x^2} \right)^{1/2} dx$$

$$= \int_{-R}^{R} 2\pi \cancel{\sqrt{R^2 - x^2}} \frac{R}{\cancel{\sqrt{R^2 - x^2}}} dx$$

$$= \int_{-R}^{R} 2\pi R \, dx$$

$$= 2\pi R x \Big|_{x=-R}^{R}$$

$$= 2\pi R^2 - 2\pi R(-R)$$

$$= \underline{4\pi R^2}$$ Surface Area of a sphere of radius R.