§11.10 Taylor & Maclauren Series.

How to find a p.s. given $f(x)$.

Assume $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a| < R$

$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$

To find c_n, $n = 0, 1, 2, \ldots$

$f(a) = c_0$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots$

$f'(a) = c_1$

$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + \cdots + (n-1)(n)c_n(x-a)^{n-2} + \cdots$

$f''(a) = 2c_2$

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$$f^{(3)}(x) = 2\cdot 3c_3 + \cdots + (n-2)(n-1)nc_n(x-a)^{n-3} + \cdots$$

$$f^{(3)}(a) = 2\cdot 3\, c_3$$

$$\vdots$$

$$f^{(n)}(a) = 2\cdot 3\cdot 4\cdots n\, c_n = n!\, c_n$$

$$\therefore\ c_n = \frac{f^{(n)}(a)}{n!}, \quad n = 0, 1, 2, \ldots$$

Taylor Series for $f(x)$ centred at a for $|x-a| < R$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$|x-a| < R.$$

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Special Case: $\underline{a=0}$

centred at $x=0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is called Maclauren Series.

Example. $f(x) = e^x$

Find Maclauren Series.

$f(x) = e^x$, $f^{(n)}(x) = e^x$

$f^{(n)}(0) = e^0 = 1$, $n = 1, 2, 3, \ldots$

$$\sum_{n=0}^{\infty} \frac{(1)x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is the Maclauren Series for e^x.

Question: If $f(x)$ has derivatives of all orders when is $f(x)$ equal to its Taylor series?

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Answer is <u>NOT ALWAYS</u>

Example: $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

has derivatives of <u>ALL</u> orders even at $x = 0$.

$$f^{(n)}(0) = 0, \quad n = 0, 1, 2, \ldots$$

(use l'Hospital's Rule).

$\therefore$ Maclauren Series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 \quad \text{for all } x$$

But $f(x) \neq 0$ if $x \neq 0$.

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When is $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$?

Define n^{th} partial sum.

$$T_n(x) = \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

n^{th} order Taylor Polynomial

i.e $T_0(x) = f(a)$ $\qquad \frac{f^{(0)}(a)}{0!} = f(a)$, $0! = 1$

$T_1(x) = f(a) + f'(a)(x-a)$

(Tangent line to $f(x)$ at $x=a$)

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

The REMAINDER.

$R_n(x) = f(x) - T_n(x)$

tells how good $T_n(x)$ approximates $f(x)$

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Thm If $\lim_{n\to\infty} R_n(x) = 0$

for $|x-a| < R$

where $R > 0$ or ∞

then

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} (f(x) - R_n(x))$$

$$= f(x) - 0$$

$$= f(x)$$

$\therefore$ $f(x)$ is equal to its Taylor Series.

Estimate $R_n(x)$.

How good is $T_n(x)$ as an estimate of $f(x)$?

Taylor's Inequality.

If $|f^{(n+1)}(x)| \le M$ for $|x-a| < d$

then $|R_n(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1}$ for $|x-a| < d$.

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Application of Taylor's Inequality.

NOTE: We showed

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$ converges for all x.

i.e. $R = \infty$

using the Ratio Test.

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$ (by the DIVERGENCE TEST)

Is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$

By the Th^n we need to sho

$$\lim_{n\to\infty} R_n = 0.$$

Taylor's Inequality

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$$|R_n(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1}$$

for $\underline{|x-a| < d}$.

where $|f^{(n+1)}(a)| \le M$.

$(a=0)$

Fix any $d \in \mathbb{R}$.

$f(x) = e^x \qquad f^{(n)}(x) = e^x$

$\max\{ |f^{(n+1)}(x)| : |x| < d \}$

$$\therefore |R_n(x)| \le \frac{e^d}{(n+1)!} |d|^{n+1} \quad \text{if } |x| < d.$$

$\to 0$ as $n \to \infty$

if $|x| < d$.

But d was arbitrary.

$\therefore$ $R_n(x) \to 0$ as $n \to \infty$

for $\underline{\text{all}}$ x

$$\therefore \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad \underline{\text{for all } x \in \mathbb{R}}$$