M1ZB3 Lecture 12 (C02) Dr. Wolkowicz Jan. 31

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§11.6 cont'd.

NOTE: Both Ratio and Root Tests fail for p-series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1/(n+1)^p}{\frac{1}{n^p}} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^p$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}} \right)^p = 1$$

TEST gives $\underline{\underline{\text{no}}}$ info.

Root Test

$$\lim_{n\to\infty} \left(\frac{1}{n^p}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{p/n}} = 1.$$

again test fails.

We used the integral test instead.

BEWARE: If the root test or the ratio test gives the limit 1, the other test will also have limit 1, and hence also fail. USE A DIFFERENT TEST.

HW need §11.7.

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§11.8. Power Series (p.s.)

Def'n A POWER SERIES p.s) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where

x is a variable

& the number c_n are called COEFICIENTS of the p.s.

Example: $\sum_{n=0}^{\infty} x^n$

$$= 1 + x + x^2 + \cdots + x^n + \cdots$$

NOTE: Convention is.

$$\begin{cases} x^0 = 1 \\ 0^0 = 1 \end{cases}$$

This is a geometric series: $\sum_{n=1}^{\infty} x^{n-1}$ which converges if $|x|<1$ & diverges if $|x|>1$.

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Example. Assume $c_n = 0$, $n > N$

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{N} c_n x^n$$

$$= c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N$$

a polynomial function of x (if $c_N \neq 0$, of degree N)

NOTE: Polynomials have a finite # of terms and no issue of convergence or divergence.

But p.s. may NOT converge.

Def'n A p.s. centred at a is

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots +$$

(At $x = a$, $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0$)

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$$\sum_{n=0}^{\infty} c_n(x-0)^n = \sum_{n=0} c_n x^n$$

$\therefore \sum_{n=0}^{\infty} c_n x^n$ is a p.s. centred at $\underline{0}$.

FACT There are only 3 mutually exclusive possibilities for $\sum_{n=0}^{\infty} c_n(x-a)^n$.

(i) The series converge ONLY at $x = a$.

(ii) The series converges for all real numbers x.

(iii) There exists a positive R, such that the series CONVERGES ABSOLUTELY if $|x-a| < R$ and DIVERGES if $|x-a| > R$.

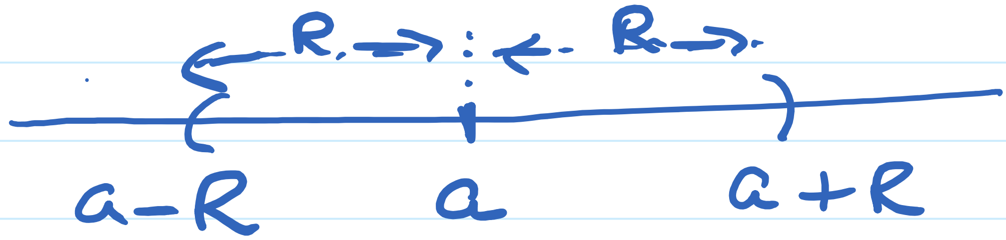
In case (iii) R is called the RADIUS of CONVERGENCE.

By CONVENTION
in case (i) $R = 0$
in case (ii) $R = \infty$

The INTERVAL of CONVERGENCE

includes $\{x : |x-a| < R\}$
and might also include
one, both, or neither
end point.

$$|x-a| < R$$
$$-R < x-a < R$$
$$a-R < x < a+R$$



End points $a-R$, $a+R$.
Each end point MUST be
considered separately.

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An interval of convergence might be

$(a-R, a+R)$ or $[a-R, a+R)$

or $(a-R, a+R]$ or $[a-R, a+R]$.

Ratio Test on $S = \sum_{n=1}^{\infty} c_n (x-a)^n$

to determine R.

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n} \right|$$

$$= \lim_{n\to\infty} \left(\left| \frac{c_{n+1}}{c_n} \right| |x-a| \right)$$

If $\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L$

then

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| |x-a| = L\,|x-a|$$

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$\therefore$ S diverges if

$$L|x-a| > 1.$$

& converges absolutely if

$$L|x-a| < 1.$$

It converges with radius of convergence R

if $|x-a| < \frac{1}{L} = R.$

(if $L \neq 0$, $L \neq \infty$)

$$\left.\begin{array}{lll} \text{if } 0 < L < \infty & & |x-a| < \frac{1}{L} = R \\ \text{if } L = 0, & & R = \infty \\ \text{if } L = \infty, & & R = 0. \end{array}\right\} (*)$$

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Similarly, using the n^{th} Root test, you obtain the same conclusion $\circledast$

BEWARE: Do not use the Ratio or Root test to test the end points to find the interval of convergence. They will fail.

Use a different TEST at EACH end point to determine the interval of CONVERGENCE