§11.6 Absolute Convergence and Ratio Test and Root Test

Def'n A SERIES $\sum a_n$ is ABSOLUTELY CONVERGENT if the series $\sum |a_n|$ converges.

Example. Alternating harmonic

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$ is <u>NOT</u> absolutely convergent even though it is convergent.

since $$\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$
the harmonic series that does NOT converge.

Example.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}, \quad p > 1$$

IS absolutely convergent.

since $\displaystyle\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{n^p} \right| = \sum_{n=1}^{\infty} \frac{1}{n^p}, \; p>1$

is convergent.

Def'n. $\sum a_n$ is CONDITIONALLY CONVERGENT if it is convergent, but NOT ABSOLUTELY convergent.

Example. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is CONDITIONALLY CONVERGENT.

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FACT The Riemann rearrangement theorem says that one can rearrange the terms of a conditionally convergent series to obtain any value including ∞.

FACT $\sum a_n$ ABSOLUTELY CONVERGENT then it is CONVERGENT. (*)

[Converse is false.
i.e. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$]

Proof of (*)
Assume $\sum a_n$ absolutely convergent.
To show $\sum a_n$ is convergent.

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$0 \le a_n + |a_n| \le 2|a_n|$

since $x + |x| = \begin{cases} 0 & \text{if } x<0 \\ 2x & \text{if } x>0. \end{cases}$

$\sum |a_n|$ convergent

$\Rightarrow \sum 2|a_n|$ convergent

$\Rightarrow \sum (a_n + |a_n|)$ is convergent by CT.

$$\underbrace{\sum(|a_n|+a_n)}_{\text{convergent}} - \underbrace{\sum |a_n|}_{\text{convergent}}$$

$$= \sum(\cancel{|a_n|} + a_n - \cancel{|a_n|})$$

$= \sum a_n$. convergent QED. χ.

(If $\sum a_n$ and $\sum b_n$ converge
$\sum(a_n - b_n) = \sum a_n - \sum b_n$.)

Ratio Test for ABSOLUTE CONVERGENCE of $\sum a_n$.

If (i) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$

$\Rightarrow \sum a_n$ CONVERGES ABSOLUTELY.

If (ii) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$

$\Rightarrow \sum a_n$ is DIVERGENT

If (iii) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$

the test fails, it is inconclusive. NO INFORMATION.

PF Idea. for (i)

Let $L < r < 1$.

Then $|a_{N+1}| < r|a_N|$ for large N.

$|a_{N+2}| < r|a_{N+1}| < r^2|a_N|$

$|a_{N+3}| < r^3|a_N|$

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$\therefore \; |a_{N+k}| \leq r^k |a_N|$

$$\sum_{i=N+1}^{\infty} |a_i| \leq \underbrace{\sum_{i=N+1}^{\infty} r^i |a_N|}$$

geometric series
convergent since $r<1$.

$\therefore \sum_{i=1}^{\infty} |a_i|$ is convergent

$\therefore \sum_{i=1}^{\infty} a_i$ is ABSOLUTELY CONVERGENT.

(ii) $|a_{n+1}| > k|a_n|$ for large n.

$\therefore \lim_{n\to\infty} a_n \neq 0$

$\sum a_n$ is divergent
by the Test for Divergence.

(iii) $L=1$ NO information
by examples

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Example. (Ratio Test).

$$S = \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)^2/2^{n+1}}{(-1)^n \; n^2/2^n}\right|$$

$$= \left(\frac{n+1}{n}\right)^2 \left(\frac{2^n}{2^{n+1}}\right)$$

$$= \left(1+\frac{1}{n}\right)^2\left(\frac{1}{2}\right) \to \frac{1}{2} < 1$$

as $n \to \infty$.

Since $L = \frac{1}{2} < 1$, S converges ABSOLUTELY by the Ratio Test.

Ratio test is especially goo for series with factorials.

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Example: $S = \sum_{n=1}^{\infty} \frac{n^n}{2^n n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1}/2^{n+1}(n+1)!}{n^n/2^n n!}\right|$$

$$= \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{2^n}{2^{n+1}}$$

$(n+1)! = n!(n+1)$

$$= \left(\frac{(n+1)}{n}\right)^n \frac{\cancel{(n+1)}\ \cancel{n!}}{\cancel{(n+1)}\ \cancel{n!}} \left(\frac{1}{2}\right)$$

$$= \left(1+\frac{1}{n}\right)^n \left(\frac{1}{2}\right) \to \frac{1}{2} e > 1.$$

$e \approx 2.718\ldots$

as $n \to \infty$.

(1^∞ is indeterminate form – use l'Hospital's Rule).

$L > 1 \Rightarrow S$ Diverges.

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Root Test for ABSOLUTE CONVERGENCE of $\sum a_n$.

(i) $\lim_{n\to\infty} |a_n|^{1/n} = L < 1$

$\Rightarrow$ $\sum a_n$ is ABSOLUTELY CONVERGENT

(ii) $\lim_{n\to\infty} |a_n|^{1/n} = L > 1$

$\Rightarrow$ $\sum a_n$ is DIVERGENT.

If $L = 1$ or limit does not exist, root test fails.

Example. $S = \sum_{n=1}^{\infty} \left(\frac{3n+4}{5n+1}\right)^n$

$$|a_n|^{1/n} = \frac{3n+4}{5n+1} = \frac{3 + 4/n}{5 + 1/n}$$

$\to \frac{3}{5} < 1$ as $n \to \infty$

$\therefore$ S converges absolutely by the root test.