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Lecture 10

§ 11.5 ALTERNATING SERIES

- terms alternate sign from pos to neg
or neg to pos.

Ex.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$= \sum_{i=1}^{\infty} (-1)^{i-1} \frac{1}{i}$$

(Alternating HARMONIC series.)

Ex.

$$-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots \quad \text{Alternating}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n-1} \quad \text{Alternating.}$$

(BUT NOT ALTERNATING)

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \dots$$

Alternating Series have form.

$$\pm \sum_{n=1}^{\infty} (-1)^{n-1} b_n, \quad b_n > 0.$$

Alternating Series TEST (AST)

Consider $\sum_{n=1}^{\infty} (-1)^{n-1} b_n, \quad b_n > 0$

with. (i) $b_n \geq b_{n+1}$ (b_n 's monotone decreasing for $n \geq \bar{n}$ for some \bar{N}).

(ii) $\lim_{n \rightarrow \infty} b_n = 0$.

Then. $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

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Example. $S = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n}\right)$ Alternating Harmonic.

$$b_n = \frac{1}{n} > 0 \quad \checkmark$$

(i) $b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1} \therefore$ decreasing. ✓

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0. \quad \checkmark$$

$\therefore S$ converges by the AST.

Example. $S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3}{n^4 + 2n + 1}$

Alternates ✓

$$b_n = \frac{n^3}{n^4 + 2n + 1} > 0$$

(i) To show $b_n \geq b_{n+1}$ i.e. b_n decreasing

Consider $f(x) = \frac{x^3}{x^4 + 2x + 1}, b_n = f(n)$

$$f'(x) = \frac{3x^2(x^4 + 2x + 1) - (4x^3 + 2)x^3}{(x^4 + 2x + 1)^2} \text{ denom} > 0.$$

$$\begin{aligned} \text{numerator} &= -x^6 + 4x^3 + 3x^2 \\ &= x^2(-x^4 + 4x + 3) = (nf)'(x) \end{aligned}$$

$$(nf)'(2) = 2^2(-16 + 8 + 3) = 2^2(-5) < 0.$$

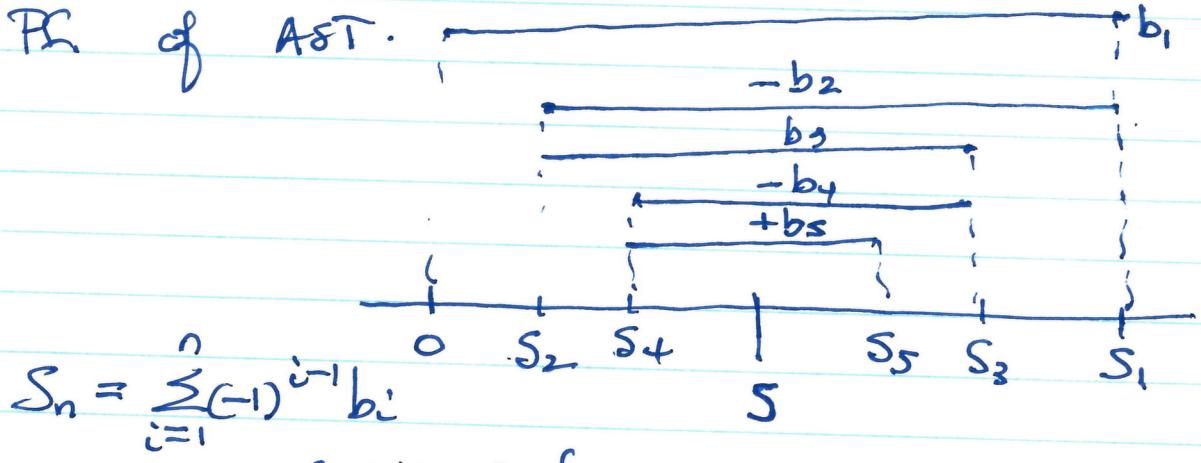
numerator < 0 if $x \geq 2$,

(ii) $\therefore b_n \geq b_{n+1}$ ✓

$$\begin{aligned} \text{(ii)} \lim_{n \rightarrow \infty} \frac{n^3}{n^4 + 2n + 1} &= \lim_{n \rightarrow \infty} \frac{1}{n + \frac{2}{n^2} + \frac{1}{n^3}} = 0. \\ \therefore S \text{ converges by AST.} \end{aligned}$$



Pf of AST.



$$S_n = \sum_{i=1}^n (-1)^{i-1} b_i$$

sequence of

Show even, partial sums, S_{2n} form an increasing sequence bounded above by b_1 .

$$S_2 = b_1 - b_2 \geq 0 \quad (b_2 \leq b_1)$$

$$S_4 = S_2 + (b_3 - b_4) \geq S_2 \quad (b_4 \leq b_3)$$

$$S_6 = S_4 + (b_5 - b_6) \geq S_4 \quad (b_6 \leq b_5)$$

$$S_{2n} = S_{2n-2} + (b_{2n-1} - b_{2n}) \geq S_{2n-2}$$

$$\therefore S_2 \leq S_4 \leq S_6 \leq \dots \leq S_{2n} \dots$$

BUT.

$$S_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \dots - (b_{2n-2} - b_{2n-1}) - b_{2n} \geq 0$$

$$\leq b_1$$

$\{S_{2n}\}_{n=1}^{\infty}$ is monotone increasing and bounded above by b_1 . \therefore it converges. Say $S_{2n} \xrightarrow{n \rightarrow \infty} S$

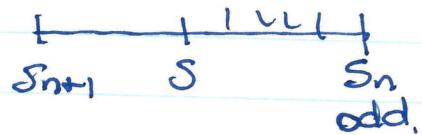
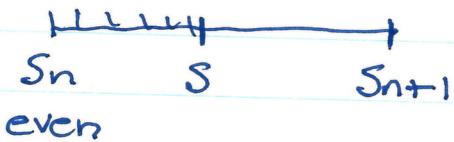
(4)

Next show $\{S_{2n+1}\}$ converges to the same thing.

$$\begin{aligned}
 \lim_{n \rightarrow \infty} S_{2n+1} &= \lim_{n \rightarrow \infty} (S_{2n} + b_{2n+1}) \\
 &= \lim_{n \rightarrow \infty} S_{2n} + \lim_{n \rightarrow \infty} b_{2n+1} \\
 &= S + 0 \\
 &= S, \text{ the same limit.} \\
 \therefore \text{QED.}
 \end{aligned}$$

Error estimate

$$\lim_{n \rightarrow \infty} S_n = S$$



$$R_n = |S - S_n| \leq |S_{n+1} - S_n| = b_{n+1}.$$

Remainder estimate for alternating series

$$\text{If } \sum_{n=1}^{\infty} (-1)^{n-1} b_n, \quad b_n > 0$$

$$(i) \quad b_n \geq b_{n+1}$$

$$(ii) \quad b_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\text{Then. } |R_n| \leq b_{n+1}.$$

Example. $S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$. How many terms do we need to

(to show $b_n > 0, b_{n+1} \geq b_n, b_n \rightarrow 0$) estimate S to AST satisfied. $\omega n \rightarrow \infty$ within 10^{-4} .

$$|R_n| \leq b_{n+1} = \frac{1}{(n+1)^2} \leq 10^{-4}$$

$$n = 99 \text{ will guarantee } (n+1)^2 > 10^4 \Rightarrow n+1 \geq 10^2 \Rightarrow n \geq 99. \quad R_n \leq 10^{-4}.$$