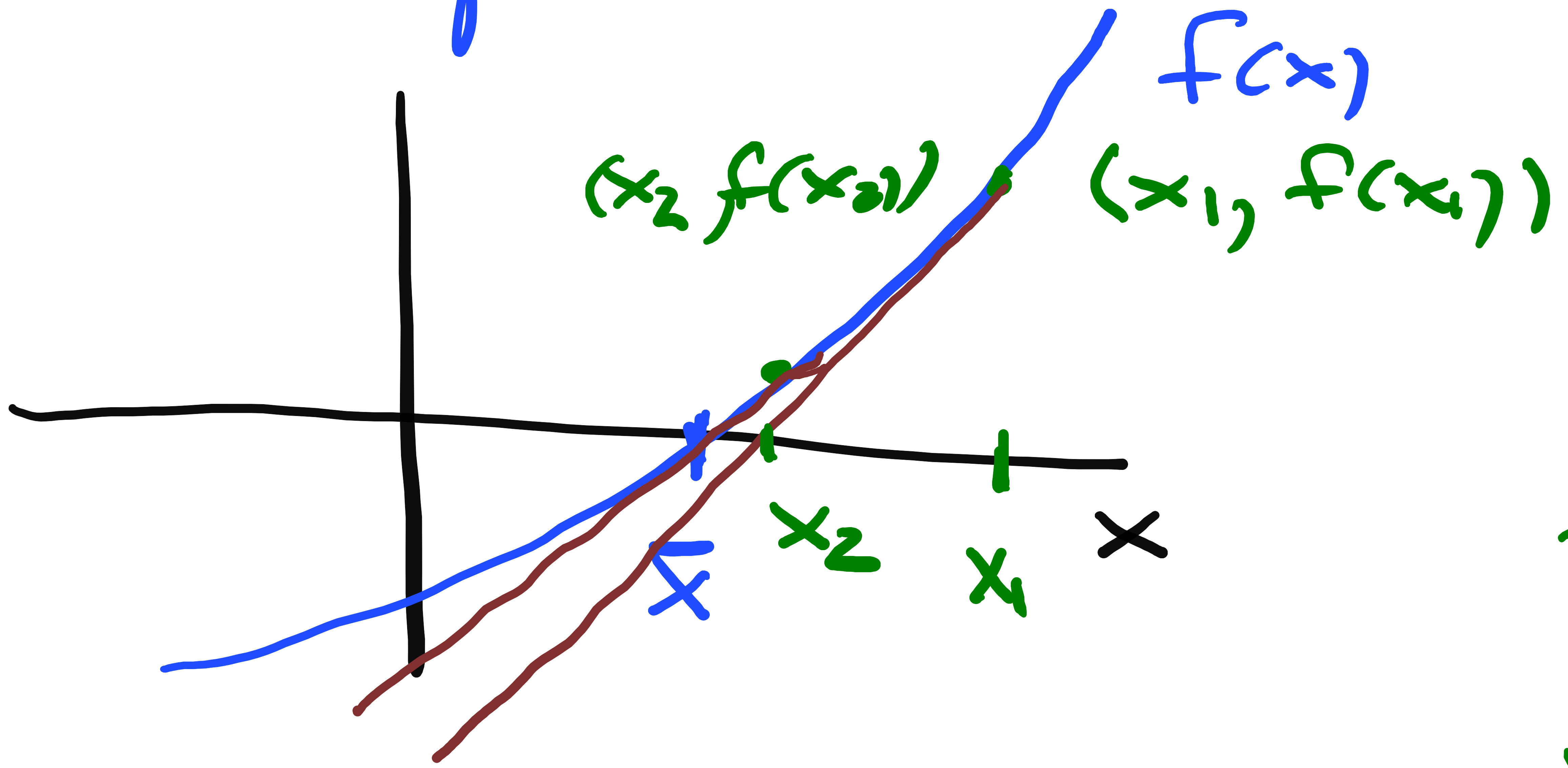
§4.8 Newton's Method.

For finding "numerical" sol'ns to $f(x)=0$ using calculators or computers.

To find $\bar{x}$ so that $f(\bar{x})=0$.



1) Initial guess: x_1

2) Find where the tangent line to the graph of f at $(x_1, f(x_1))$ intersects the x-axis, and call that point x_2.

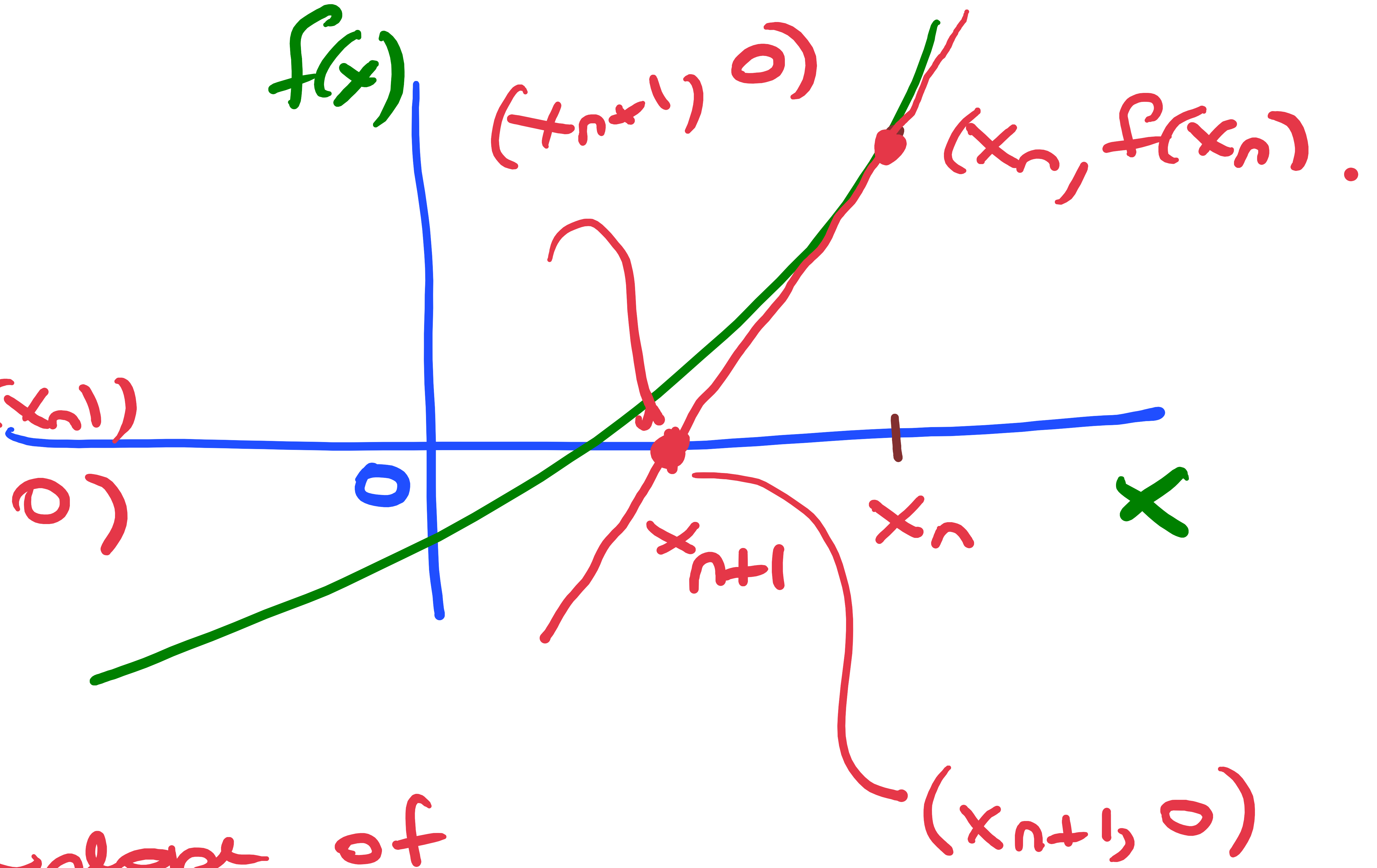
REPEAT from x_2.

If x_1 is "close enough" and f is diff'ble we generate a sequence

$$\{x_1, x_2, x_3, \ldots, x_n, \ldots\}.$$

and $x_n \to \bar{x}$, soln of $f(x)=0$, as $n \to \infty$.

What is the formula to generate x_{n+1} from x_n.



Equation of the tangent line: is the line through the points $(x_n, f(x_n))$, $(x_{n+1}, 0)$

$$\frac{0 - f(x_n)}{x_{n+1} - x_n} = f'(x_n)$$

slope of the tangent line at $(x_n, f(x_n))$.

Solve for x_{n+1}

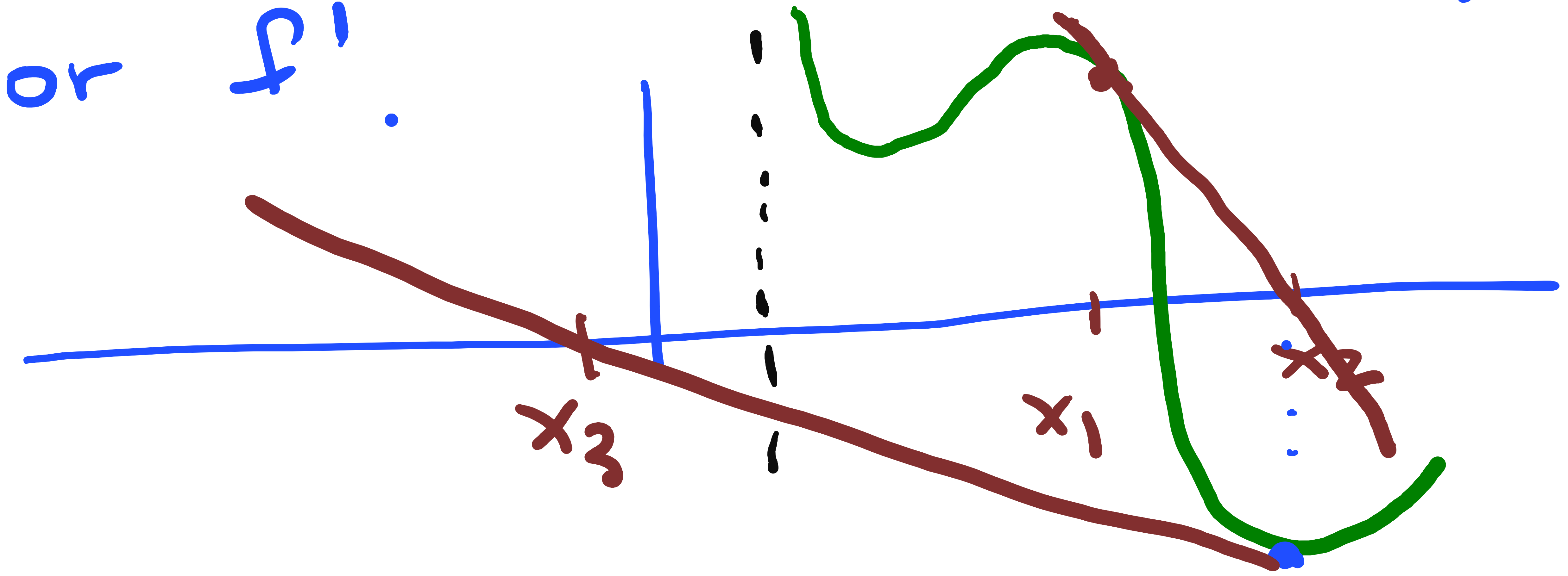
$$0 - f(x_n) = (x_{n+1} - x_n) f'(x_n),$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad n = 1, 2, 3, \ldots$$

Newton's Method

What can go wrong?

(1). One of the iterates might fall outside the domain of f or f'.

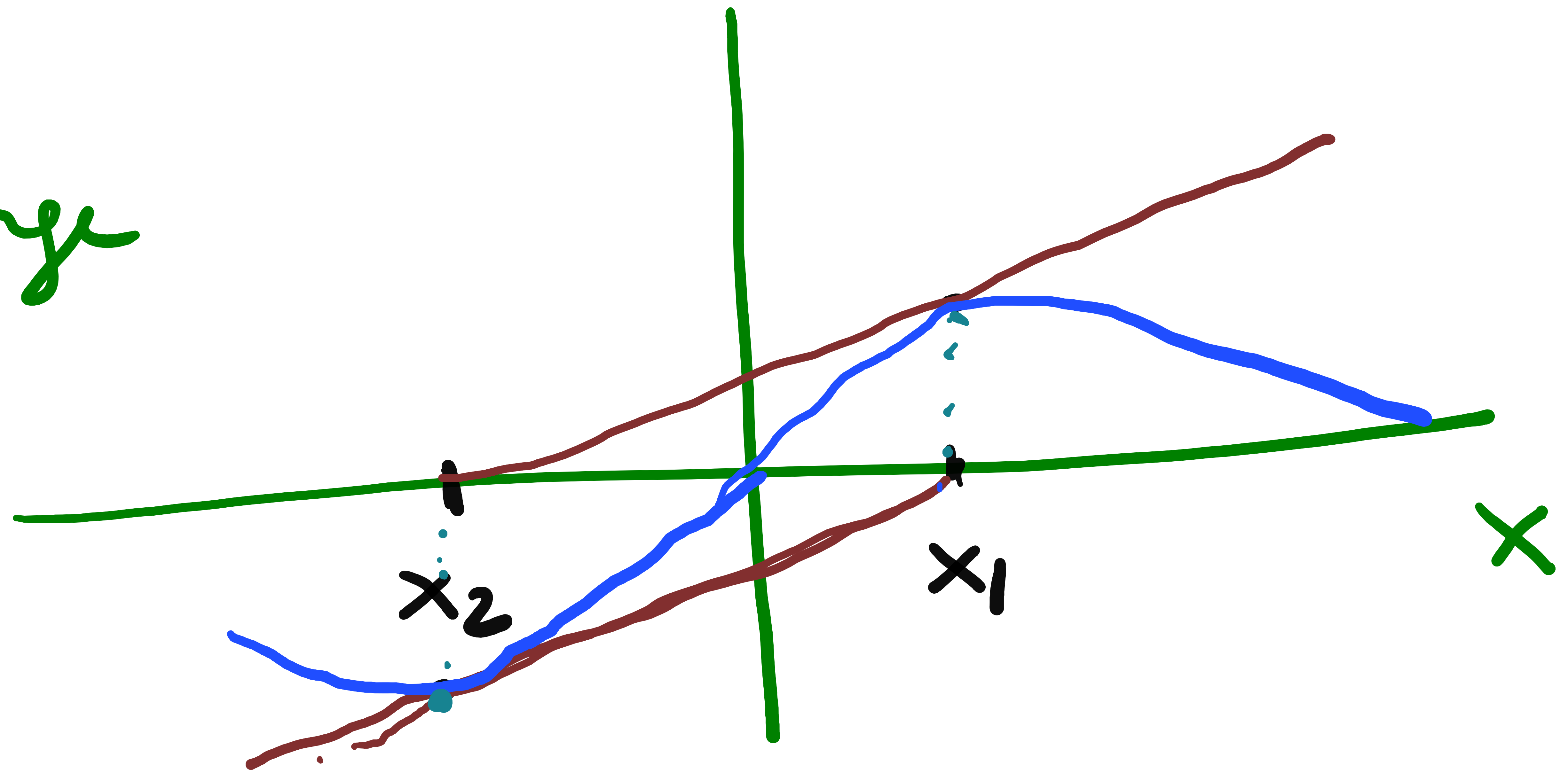


x_3 is NOT in the domain of f.

(2). Might cycle and not converge to a root.

$x_1, x_2, \overset{x_1}{\overset{\parallel}{x_3}}, \overset{x_2}{\overset{\parallel}{x_4}}, \ldots$

Sequence does NOT converge.



Estimate of accuracy of sol'n.
i.e., knowing when to STOP.

Rule of thumb:

Stop when successive iterates (i.e., successive approximations) x_n and x_{n+1} agree (are the same) when rounded to the number of decimal places of accuracy required.

If you want 3 decimal places of accuracy, then you want x_n and x_{n+1} to agree to 3 decimal places. When this happens, STOP.

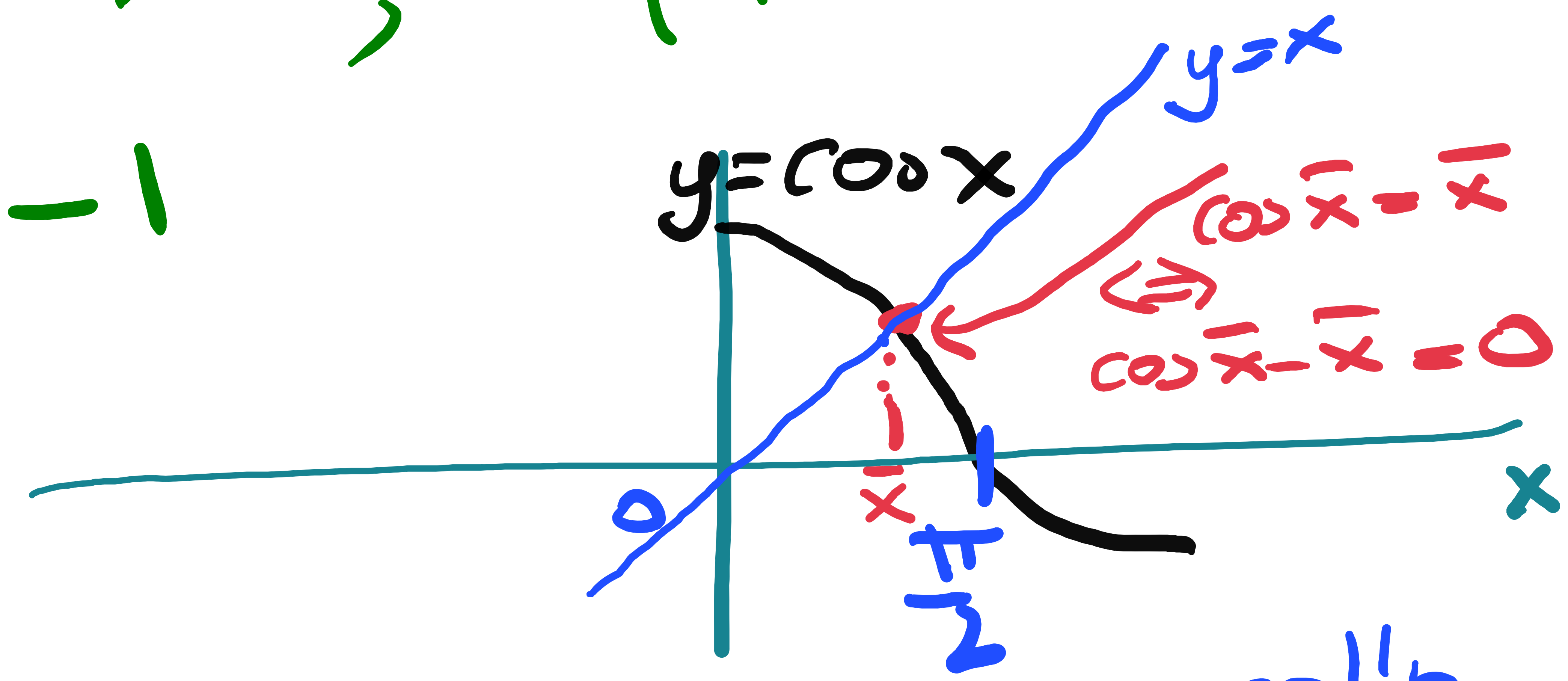
Example Find correct to 3 decimals the solution of the equation: $\cos x = x$.

Note: $\cos x = x$ iff $\cos x - x = 0$.

Sol'n: Newton's Method.

Let $f(x) = \cos x - x$, $\{$When $f(x)=0$, then $\cos x = x$.$\}$.

$f'(x) = -\sin x - 1$



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\boxed{x_{n+1} = x_n + \frac{(\cos x_n - x_n)}{+\sin x_n + 1}}.$$

$n = 1, 2, 3, \dots$

I multiplied numerator and denominator by (−1) to change − to +

There is a sol'n for some $x \in (0, \frac{\pi}{2})$

Try $x_1 = 1 \in (0, \frac{\pi}{2})$ as an initial guess.

$x_1 = 1$ Initial Guess.

(Note: This is not the only reasonable guess.)

$$x_2 = 1 + \frac{\cos(1) - 1}{\sin(1) + 1} = 0.750364$$

$$x_3 = x_2 + \frac{\cos(x_2) - x_2}{\sin(x_2) + 1} = 0.739113$$

$$x_4 = x_3 + \frac{\cos(x_3) - x_3}{\sin(x_3) + 1} = 0.73908\ldots$$

x_3 and x_4 agree (are equal) rounded to 3 decimal places.

∴ a sol'n correct to 3 decimal places is $x = 0.739$

§3.4. The Chain Rule.

$$\left\{\begin{array}{l} \frac{d}{dx} f(g(x)) = (f(g(x)))' \\ \qquad\qquad = f'(g(x))g'(x) \\ \\ (f\circ g)'(x) = f'(g(x))g'(x). \end{array}\right.$$

Ⓘ.

$$f(g(h(x))) = f'(g(h(x))g'(h(x))h'(x)$$

In Leibniz notation.

$f(g(x))$. Let $u = g(x)$

$y = f(g(x)) = f(u)$

then $$\boxed{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}}$$ II

$f(g(h(x)))$

Let $h(x) = v$

$g(h(x)) = w$

$y = f(g(h(x))$

$= f(g(v))$

$= f(w).$

$$\frac{dy}{dx} = \frac{dy}{dw}\,\frac{dw}{dv}\,\frac{dv}{dx}$$

Example

$y = \cos(x^2)$

$= f(g(x))$ where $g(x) = x^2$

$f(x) = \cos(x)$

Find $\frac{dy}{dx} = (f(g(x)))'$:

Ⓘ $$\frac{dy}{dx} = f'(g(x))\,g'(x)$$

$$= -\sin(\underbrace{x^2}_{g(x)})\,2x$$

OR using Leibniz notation:

Ⅱ Let $u = x^2$, $\therefore\ y = \cos(u)$ $\quad (y = \cos(x^2))$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$= -\sin(\underset{\underset{u}{\uparrow}}{x^2})\, 2x$$

same result