§3.1. Exponential fcns. $\frac{d}{dx} b^x$. $f(x) = b^x$

$$\frac{d}{dx} b^x = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$

$b^{x+h} = b^x b^h$

$$= \lim_{h \to 0} \frac{b^x (b^h - 1)}{h}$$

$$= b^x \lim_{h \to 0} \left(\frac{b^h - 1}{h} \right) = b^x f'(0).$$

$$f'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h}$$

Def'n of number e

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$$\Rightarrow \boxed{\frac{d}{dx} e^x = e^x}$$

§3.2 Product & Quotient Rule

Product Rule (f & g are both diff'ble)

then $\frac{d}{dx}\left(f(x)g(x)\right) = \frac{df(x)}{dx}\, g(x) + f(x)\frac{d}{dx}g(x)$

Example

$$\frac{d}{dx}\left(f(x)g(x)h(x)\right) = \frac{df}{dx}\, g(x)h(x) + f(x)\frac{dg}{dx}h(x) + f(x)g(x)\frac{dh}{dx}$$

Example: Find $h''(x)$ if $h(x) = \underbrace{x^2}_{f}\underbrace{e^x}_{g}$

$$h'(x) = \underbrace{2xe^x}_{} + \underbrace{x^2e^x}_{}$$

$$h''(x) = \underbrace{2e^x + 2xe^x}_{} + 2xe^x + x^2e^x$$

$$= e^x\left(x^2 + 4x + 2\right)$$

Quotient Rule

f, g diff'ble, $g(x) \neq 0$.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx} g(x) - \frac{dg}{dx} f(x)}{(g(x))^2} \quad \text{where } g(x) \neq 0.$$

Example. $y(x) = \dfrac{x^3-4}{\sqrt{x}}$, $x>0$

$$y'(x) = \frac{3x^2\sqrt{x} - \frac{1}{2}x^{-1/2}(x^3-4)}{x}$$

$$\text{or } (\sqrt{x})^2$$

HW $\vdots$

$$= \frac{5x^3+4}{2x^{3/2}}.$$

$\sqrt{x} = x^{1/2}$

$\dfrac{d\sqrt{x}}{dx} = \dfrac{1}{2}x^{-1/2}$

$\dfrac{dx^n}{dx} = nx^{n-1}$

$n = 1/2$.

§3.3. Derivative of Trig Functions.

To show $\frac{d}{dx}\sin(x) = \cos x$ from the def'n.

We will need to show first

(a) $\lim\limits_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$, (b) $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

(a). $\lim\limits_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = \lim\limits_{\theta \to 0} \left(\frac{\cos\theta - 1}{\theta}\right)\underbrace{\left(\frac{\cos\theta + 1}{\cos\theta + 1}\right)}_{=1}$

$= \lim\limits_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \lim\limits_{\theta \to 0} \frac{-\sin^2\theta}{\theta(\cos\theta + 1)}$

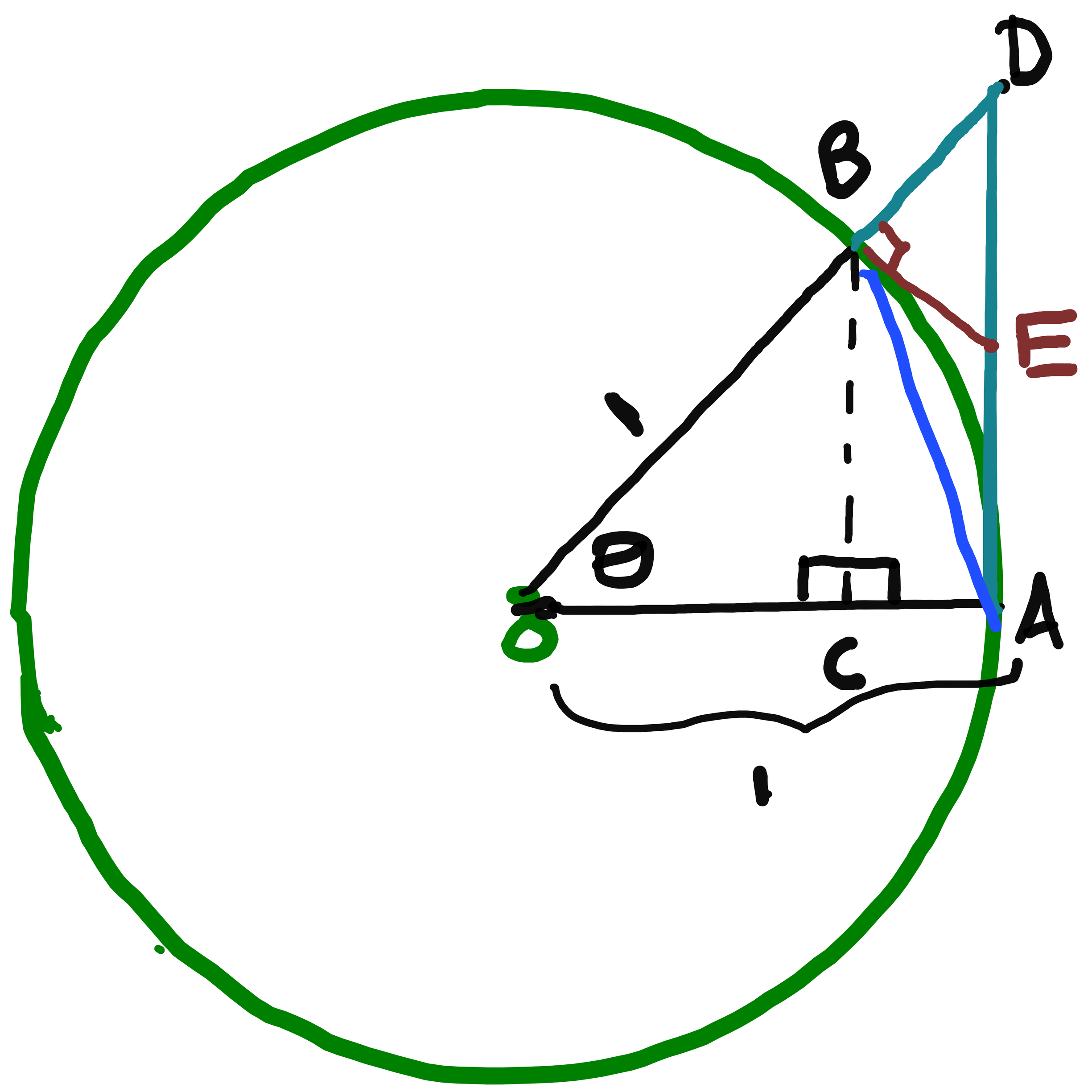
$= -\lim\limits_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin\theta}{\cos\theta + 1}\right) = -\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} \lim\limits_{\theta \to 0} \frac{\sin\theta}{\cos\theta + 1}$

$= -(1)(0) = 0$

$\boxed{\therefore \lim\limits_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0}$ $\longrightarrow$ provided we prove $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Next we show $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

(b) Let $0 < \theta < \frac{\pi}{2}$.



$|OB| = 1$ and $|OA| = 1$

$\text{arc } AB = \theta$

$\sin\theta = \frac{|BC|}{|OB|} = \frac{|BC|}{1}$

$\Rightarrow |BC| = \sin\theta$

$\sin\theta = |BC| < \underset{\text{hypotenuse}}{|AB|} < \text{arc } AB = \theta$

$\therefore \sin\theta < \theta$

$\Rightarrow \boxed{\frac{\sin\theta}{\theta} < 1}$ (*)

Also,

$\theta < \text{arc } AB < |AE| + |BE| < |AE| + \underset{\text{hypotenuse}}{|DE|} = |AD| = \underset{\substack{\parallel \\ 1}}{|OA|} \tan\theta = \tan\theta$

$\therefore \quad \theta < \tan\theta$

$\iff \theta < \frac{\sin\theta}{\cos\theta} \iff \cos\theta < \frac{\sin\theta}{\theta}$

But we showed (see (*) above) $\frac{\sin\theta}{\theta} < 1$

$\therefore \quad \underset{1 \text{ as } \theta \to 0}{\cos\theta} < \frac{\sin\theta}{\theta} < \underset{1 \text{ as } \theta \to 0}{1}$

By the Squeeze Thm

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.}$$

Now we use the def'n of the derivative to prove that $\frac{d}{dx} \sin x = \cos x$.

$$\frac{d}{dx} \sin(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \cos x \frac{\sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \frac{(\cos h - 1)}{h} \right) + \lim_{h \to 0} \cos x \frac{\sin h}{h}$$

$$\frac{d}{dx}\csc x = \frac{d}{dx}\frac{1}{\sin x} = \frac{0-\cos x}{\sin^2 x} = -\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin x}\right)$$

$$= -\cot x \csc x$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{+\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x \sec x$$

Note: Can always write $\tan x$, $\cot x$, $\sec x$, $\csc x$ in terms of $\cos x$ and/or $\sin x$ and then differentiate using quotient rule or product rule.