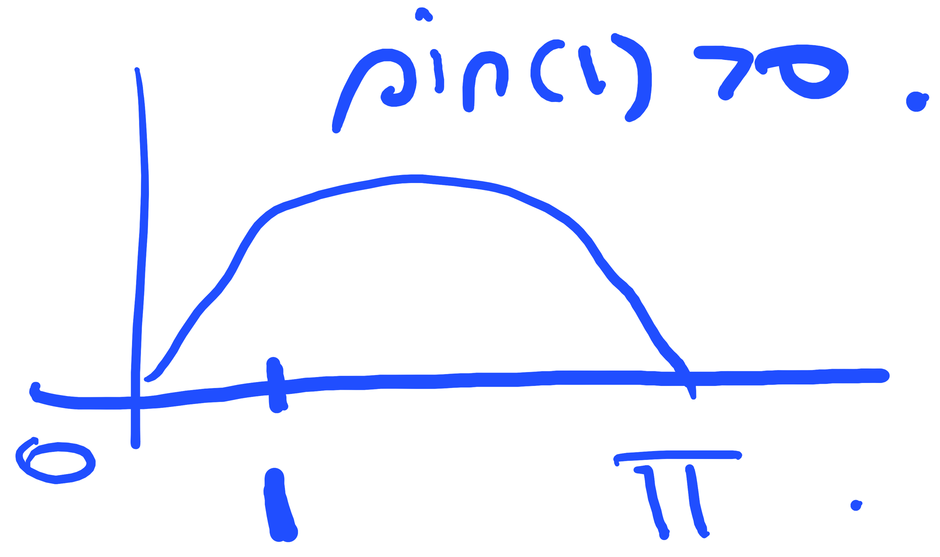
Intermediate Value Thm (IVT)

Example: Show that there is a solution of the equation $\sin(x) = x^2 - x$, $x \in (1,2)$

Sol'n Let $f(x) = \sin(x) - x^2 + x$

There is a sol'n of our original equation if there is an $\bar{x}$ such that $\begin{cases} f(\bar{x}) = 0, \\ \bar{x} \in (1,2). \end{cases}$

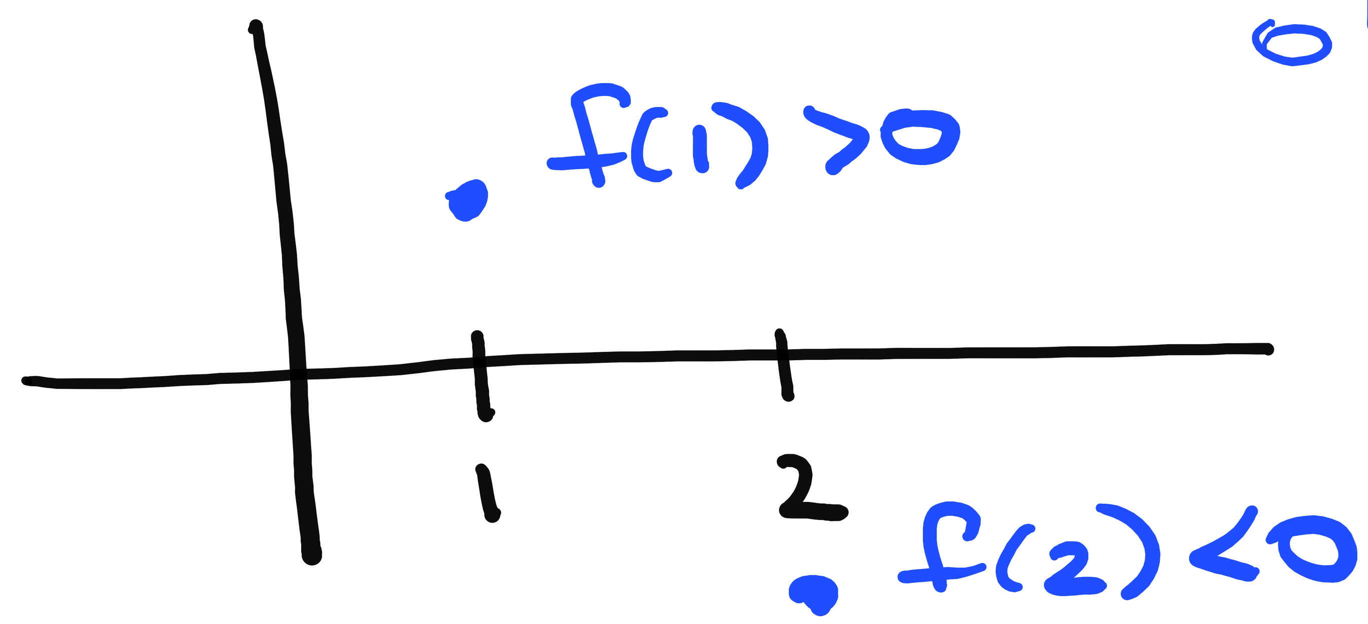
Note, $f(x)$ is a continuous function on $[1,2]$.



$$f(1) = \sin(1) - 1^2 + 1 = \sin(1) > 0$$

$$f(2) = \sin(2) - 2^2 + 2$$

$$= \sin(2) - 2 < 0.$$



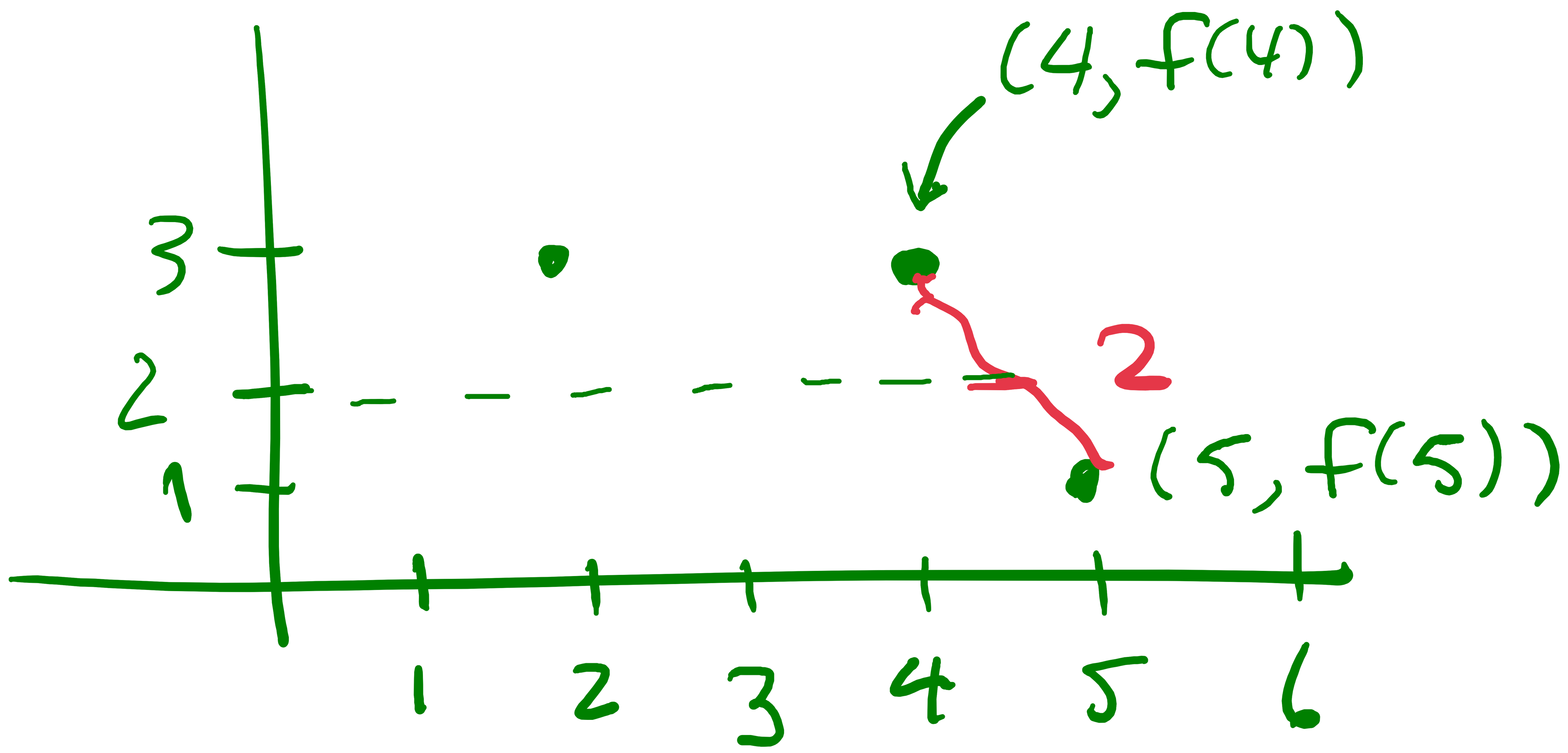
$f(1) > 0$ and $f(2) < 0$

and so $N = 0$ is between $f(1)$ and $f(2)$, by the IVT there is a $c \in (1, 2)$ such that $f(c) = 0$.

i.e., $x = c$ is a sol'n of the original problem in $(1, 2)$.

Example Let $f(x)$ be continuous on $[0, 6]$.

Given
$f(2) = 3$
$f(4) = 3$
$f(5) = 1$



(a) Can you say whether or not there is an $a \in (4, 5)$ such that $f(a) = 2$? YES. By the IVT.

$f(4) = 3$, $f(5) = 1$, and $N = 2$ is between $f(4)$ and $f(5)$.

(b) Can you say whether or not there is an $a \in (2,4)$ such that $f(a) = 2$?

No! IVT fails to tell us.

HW Review Limit Laws (§2.3)

Continuity Laws

$f(x)$ is continuous at $x=a \iff \lim_{x\to a} f(x) = f(a)$.

$g(x)$ " " " " $\iff \lim_{x\to a} g(x) = g(a)$.

Then, $f(x) \pm g(x)$; $f(x) \cdot g(x)$ and

$\left(f(x)/g(x) \text{ provided } g(a) \neq 0 \right)$

are continuous functions at $x = a$.

Functions that are continuous at all x in their domain include:

polynomials : $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

e.g. $3x^2 + 6x + 2$

$n = 2$, $a_2 = 3$, $a_1 = 6$, $a_0 = 2$.

rational functions : $f(x)/g(x)$ where f and g are polynomials and $g \neq 0$.

root functions

trig functions & inverse trig functions

exponential functions

logarithmic functions

" combinations of continuous functions using continuity laws "

Example: $h(x) = \dfrac{(x-1)\ln(x)}{x-4}$.

Where is h continuous?

Sol^n: $x-1$ is continuous on $\mathbb{R}$

$\ln(x)$ " " " $(0,\infty)$

$\therefore$ $(x-1)\ln(x)$ is continuous on $(0,\infty)$

$x-4$ is continuous on $\mathbb{R}$

$x-4=0$ iff $x=4$.

(iff means $\Leftrightarrow$ means if and only if)

$\therefore$ $\dfrac{1}{x-4}$ is continuous on $(-\infty,4)\cup(4,\infty)$

OR alternatively $x\in\mathbb{R}$ and $x\neq 4$.

$\therefore$ h is continuous if $x\in(0,\infty)$ and $x\neq 4$.

or alternatively

if $x\in (0,4)\cup(4,\infty)$.

Composition of functions:

$$(f \circ g)(x) = f(g(x))$$

" f of g of x "

Example: $\sqrt{x^2-1} = (f \circ g)(x)$ where $g(x) = x^2-1$
and $f(x) = \sqrt{x}$.

since $(f \circ g)(x) = f(g(x)) = f(x^2-1) = \sqrt{x^2-1}$.

BEWARE: $(f \circ g)(x) \neq (g \circ f)(x)$, in general.

HERE, $g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2-1 = x-1$.

Thm/ If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$,

i.e., $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$, (if $\lim_{x \to a} g(x)$ exists.)

Example: Find $\lim\limits_{x\to 2} \tan^{-1}\left(\dfrac{x^2-2x-2}{x-2}\right)$.

<u>Sol'n.</u> $f(x) = \tan^{-1}(x)$ is continuous for all $x \in \mathbb{R}$.

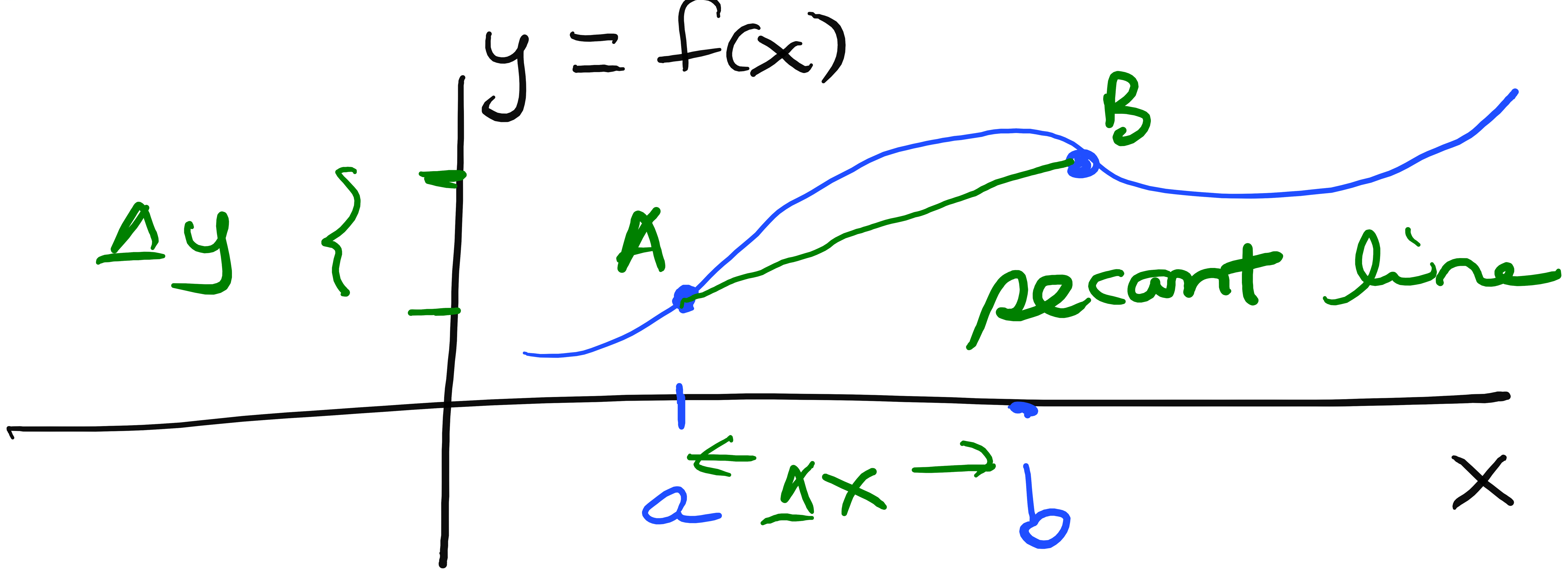
Let $g(x) = \dfrac{x^2-2x-2}{x-2}$.

$\lim\limits_{x\to 2} g(x) = \lim\limits_{x\to 2} \dfrac{x^2-2x-2}{x-2} = 3.$ (See last lecture).

$$\therefore \lim_{x\to 2} f(g(x)) = f\left(\lim_{x\to 2} g(x)\right)$$
$$= f(3)$$
$$= \tan^{-1}(x).$$

Thm/ If g is continuous at a and f is continuous at $g(a)$, then the composition $(f\circ g)(x)$ is continuous at a.

§2.7 Derivatives & Rate of Change



from $(a, f(a)) = A$

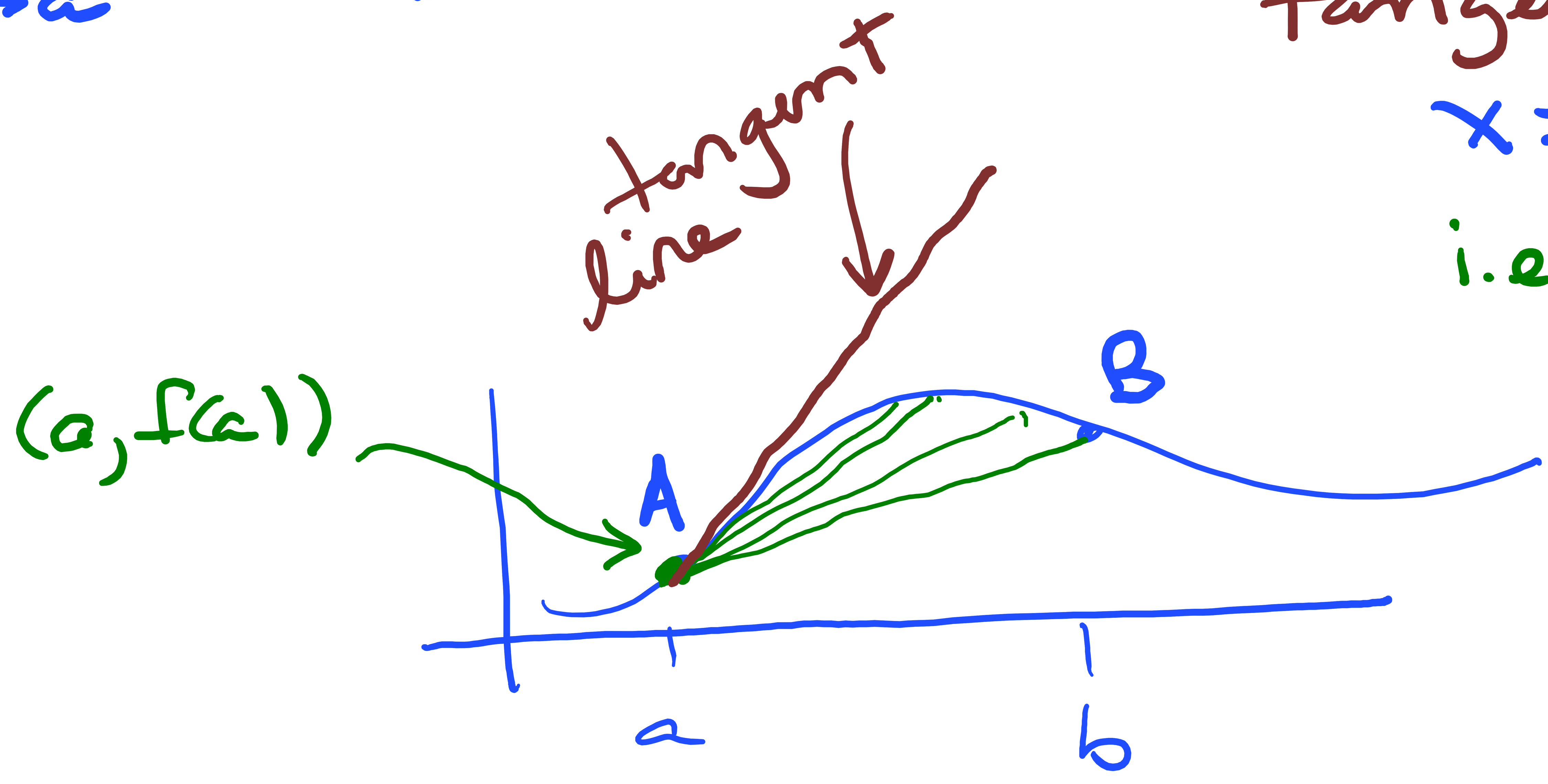
to $(b, f(b)) = B$

Slope of the <u>secant line</u> $= \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta y}{\Delta x}$

$= \dfrac{f(b)-f(a)}{b-a}$.

$\lim\limits_{b \to a} \dfrac{f(b)-f(a)}{b-a}$ = slope of the tangent line to f at $x=a$.

i.e. the line parallel to the curve passing through the point $(a, f(a))$.



Def'n The derivative of a function f at a number a is denoted $f'(a)$ (or $\frac{df(a)}{dx}$) and

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if the limit exists.

NOTE: Let $x = a+h \Rightarrow h = x-a$.

$h \to 0 \Leftrightarrow x \to a$

$$\therefore f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{x \to a} \frac{f(a+(x-a)) - f(a)}{x-a} \quad (\text{subst } h = x-a)$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a}.$$

$$\therefore \boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \begin{array}{l}\text{Slope of the} \\ = \text{tangent line} \\ \text{at } x = a.\end{array}}$$

Example. Find the equation of the tangent line to $y = x^2$ at the point $(2, 4)$.

HW Try this question. We will do it at the begining of next class.