Putnam competition

- The William Lowell Putnam competition is a university-level mathematics competition held annually for undergraduate students at North American universities. More information can be found at www.math.mcmaster.ca - follow the Undergraduate and then the Putnam competition links.
- This year's competition will occur on Dec. 7. If you are interested in participating or learning more, send email to Bradd Hart, hartb@mcmaster.ca or David Earn, earn@math.mcmaster.ca
- There will be an information session on Thursday, Sept. 12 at 11:30 in HH312.

Math 1A03 Lecture 4 Sept. 10/19

§1.5 cont'd.

Example: Solve $\ln(x^2-1)=2$

Sol'n: $e^{\ln(x^2-1)} = e^2$, $|x|>1$.

$x^2-1 = e^2$

$x^2 = e^2-1$

$x = \pm\sqrt{e^2-1}$

Changing base:

$b>0,\ b\neq 1$ $\log_b$ $a>0,\ a\neq 1$ $\log_a$

If $y=\log_b x$ what is y using $\log_a$?

Note: $b^y = b^{\log_b x} = x$ (since $f(x)=\log_b x$, $f^{-1}(x)=b^x$, $f^{-1}(f(x))=x$).

$b^y = x$

$\log_a b^y = \log_a x$

$y\log_a b = \log_a x$

$$\boxed{\log_b(x) = y = \log_a x / \log_a b}$$

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

Special Case:

e.g., $\log_b x = \dfrac{\ln x}{\ln b}$ $\quad (a=e)$.

Example $y = f(x) = x + e^x$

Does $f^{-1}(x)$ exist? YES, both x, e^x are increasing
sum is increasing.
$\therefore$ $f(x)$ is one-to-one
$\therefore$ $f^{-1}(x)$ exists.
domain of $f(x)$:
$x \in \mathbb{R}$.
range of $f(x)$:
$y \in \mathbb{R}$.

Find (a) $f^{-1}(f(25)) = 25$.

(b). $f^{-1}(1)$

It is too hard to find $f^{-1}(x)$ for all $x \in \mathbb{R}$.

But all we need is $f^{-1}(1)$.

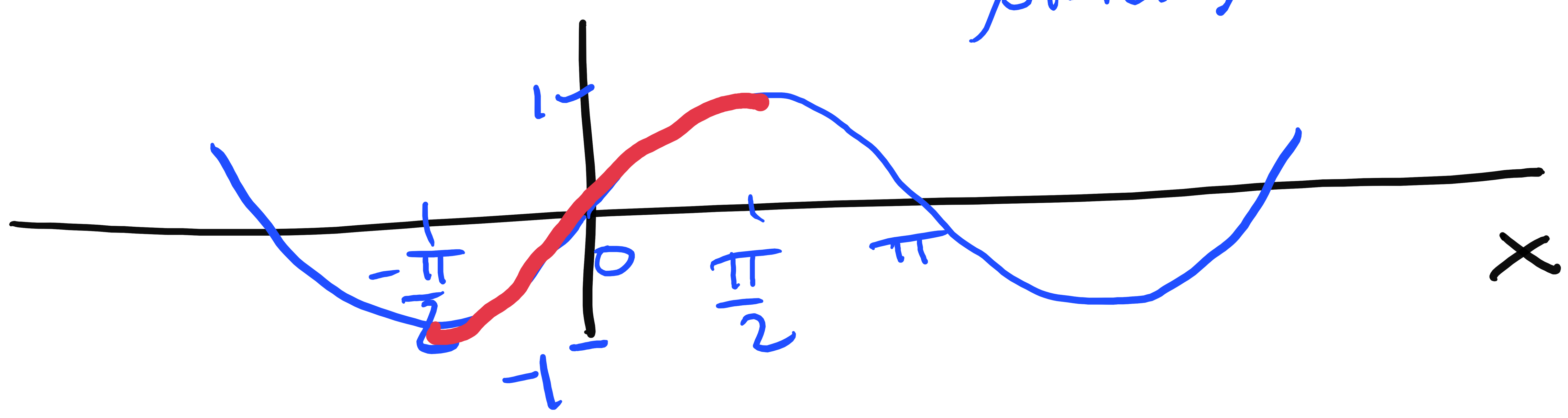
$a = f^{-1}(1)$ and find a.

$f(a) = 1$

$a + e^a = 1$. See $a = 0$ works.

$\therefore$ $f^{-1}(1) = 0$.

Inverses of Trig functions.



NOT one-to-one.

Restrict the domain of $\sin(x)$ to $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Then $\sin(x)$ IS one-to-one, and the inverse exists.

$$\sin^{-1}(x) = \arcsin(x)$$

domain: $[-1, 1]$

range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

BEWARE:

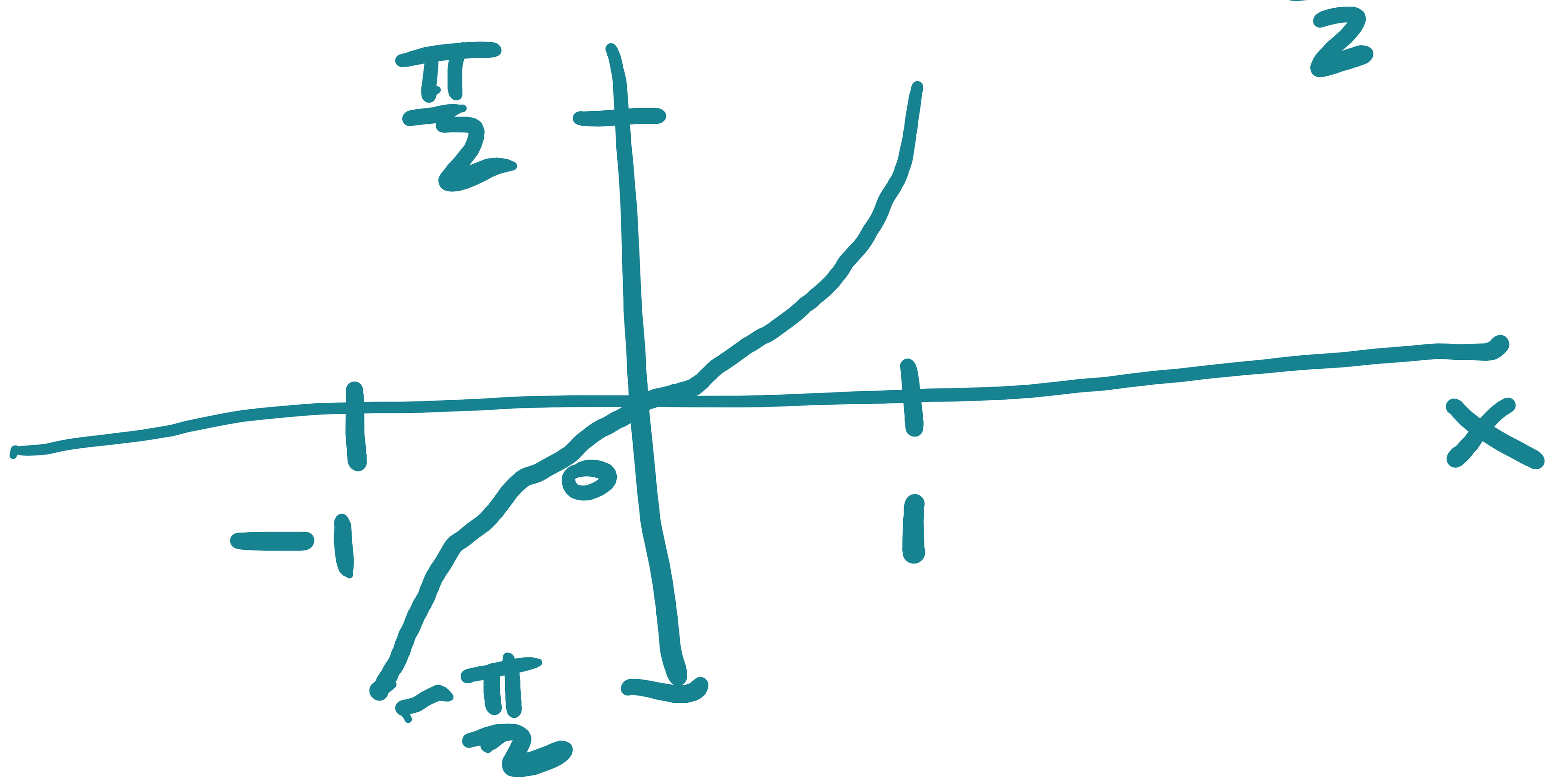
$$\sin^{-1}(x) \neq \frac{1}{\sin(x)} = (\sin(x))^{-1}$$

But

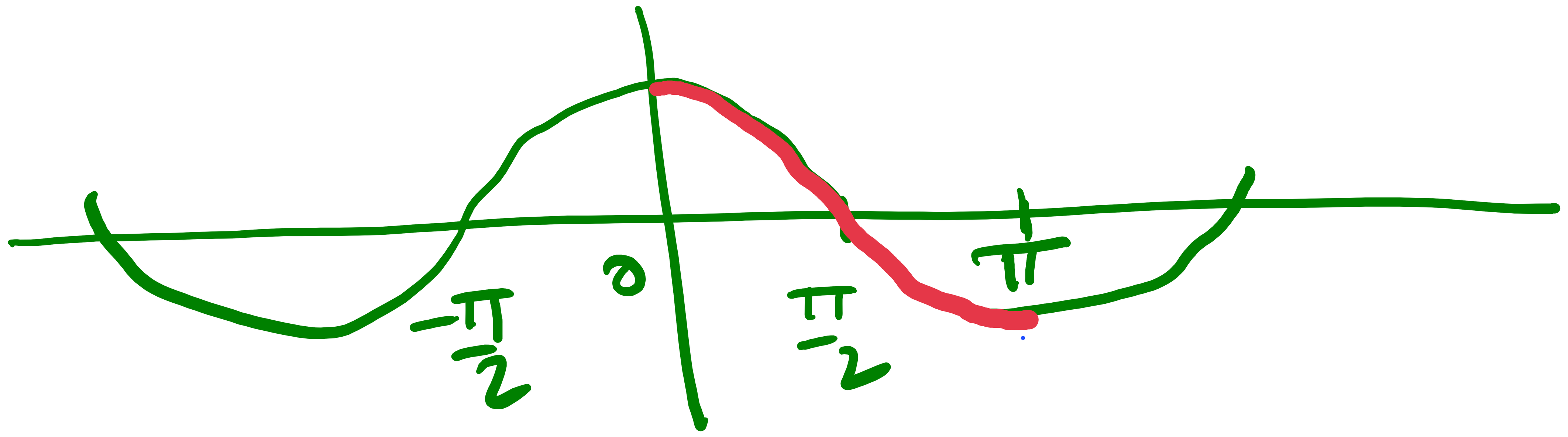
$$\sin^n(x) = (\sin(x))^n$$

if n positive integer.

$$\sin^{-1}(x) = y \iff \sin(y) = x$$
$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$



$y = \cos(x)$



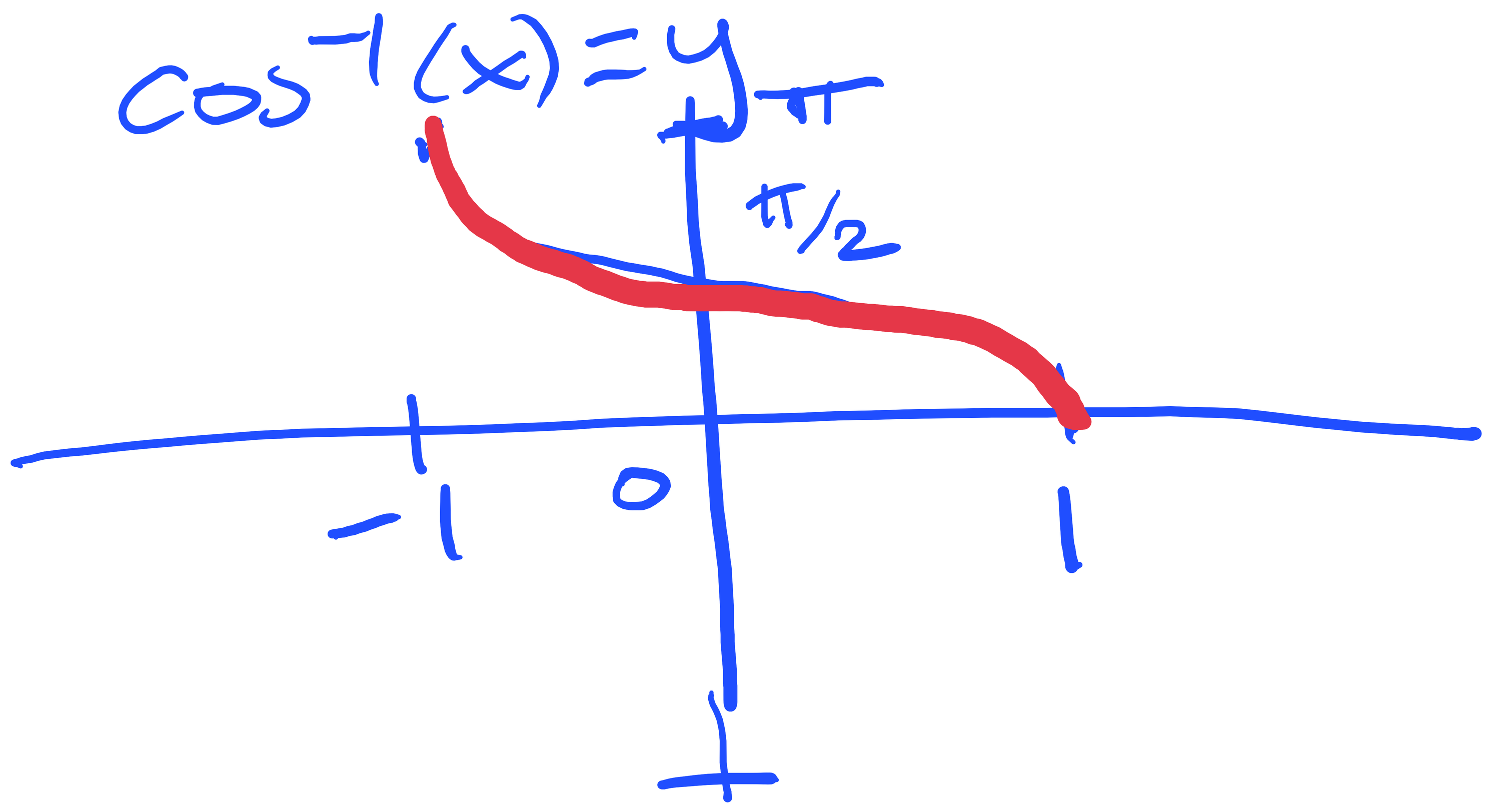
restricted $y = \cos(x)$

$x \in [0, \pi]$ closed interval

is one-to-one

$\therefore$ has inverse.

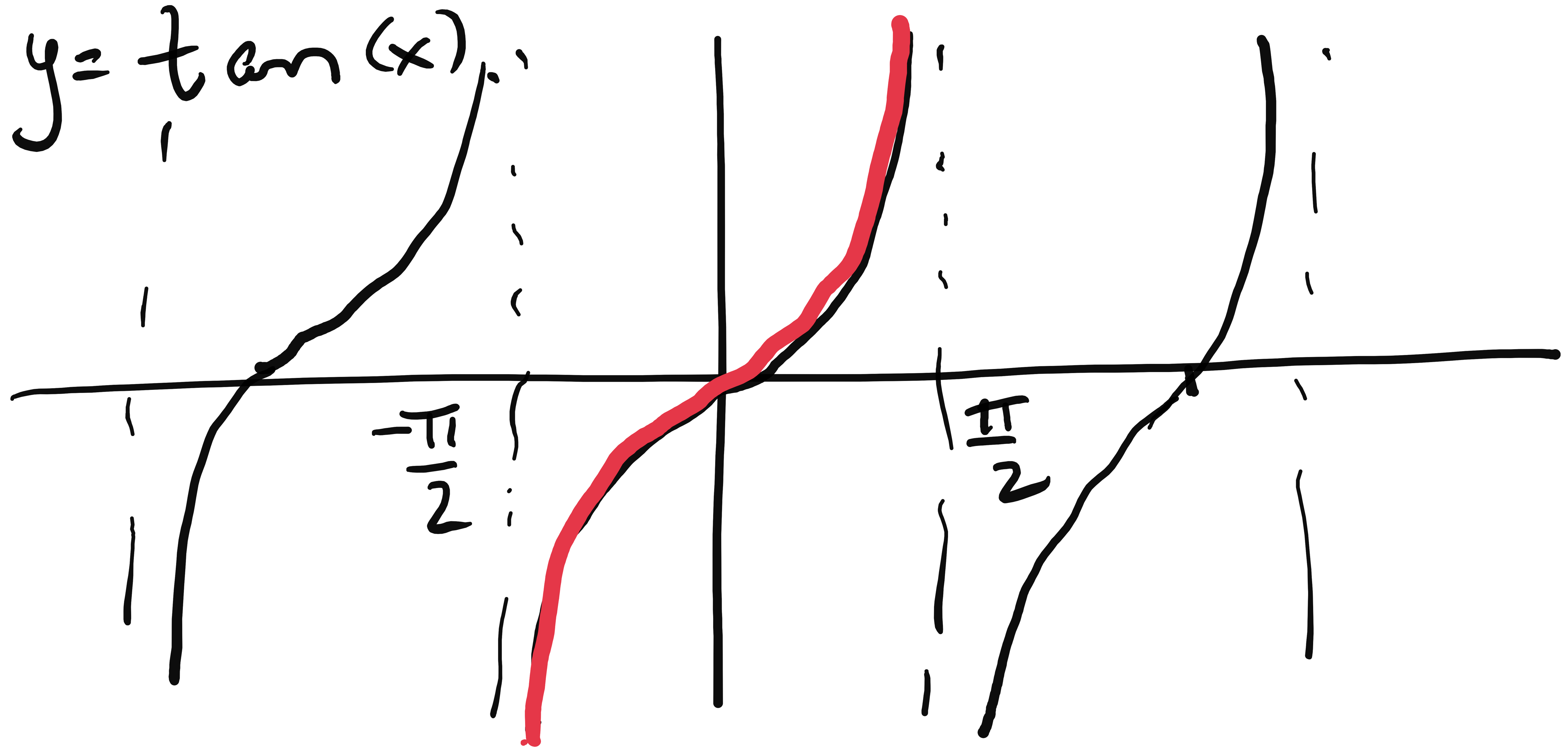
$\cos^{-1}(x) = \arccos(x)$.



$\cos^{-1}(x) = y$ iff $\cos(y) = x$ &

$y \in [0, \pi]$.

$x \in [-1, 1]$ domain, $y \in [0, \pi]$ range.



$y = \tan x$ $\quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

open interval.

$\tan^{-1}(x) = y$ iff $\tan(y) = x$

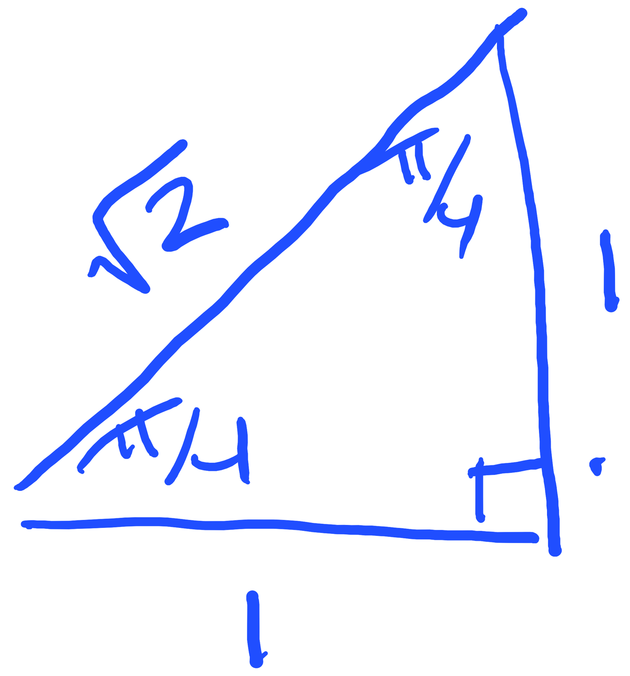
and

$-\frac{\pi}{2} < y < \frac{\pi}{2}$.

domain: $\mathbb{R}$

range: $(-\pi/2, \pi/2)$.

Example: $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \in [0,\pi]$



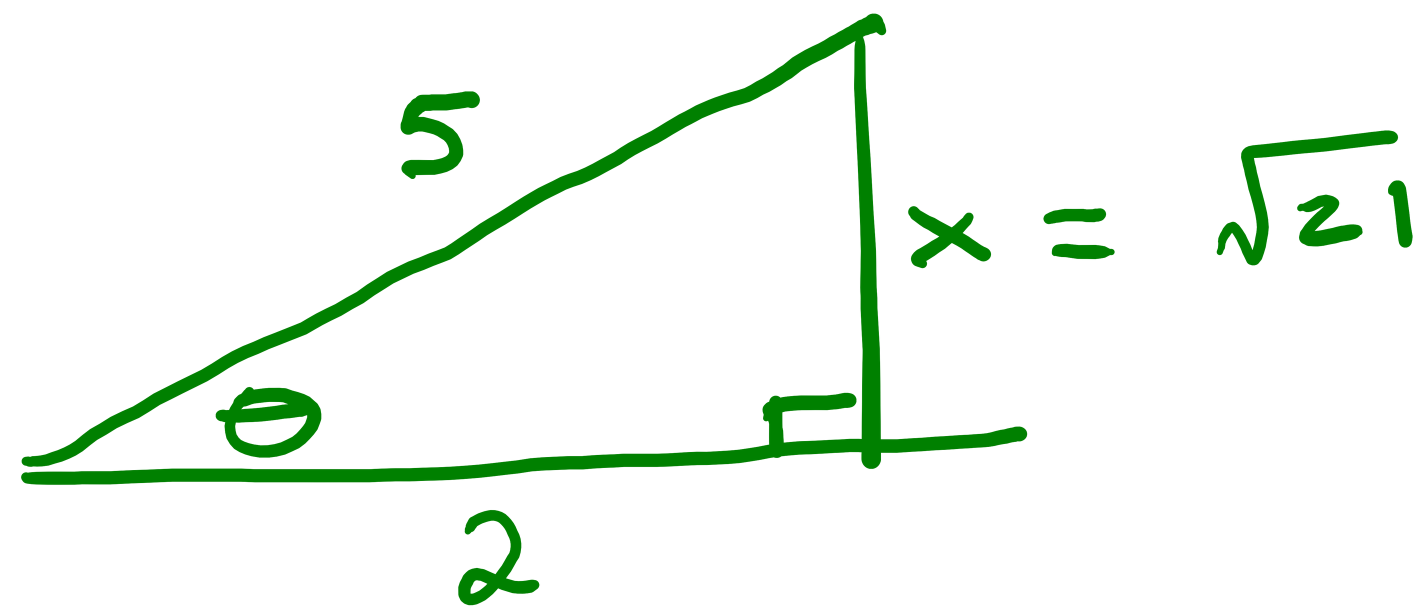
$\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

But $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \neq -\frac{\pi}{4}$ since $-\frac{\pi}{4} \notin [0,\pi]$.

Example: Evaluate $\tan\left(\cos^{-1}\left(2/5\right)\right)$.

Sol'n: Find $\tan(\theta)$ where $\theta = \cos^{-1}(2/5)$

$\cos\theta = 2/5$



$2^2 + x^2 = 5^2$

$x^2 = 25 - 4 = 21$

$x = \sqrt{21}$

$$\boxed{\therefore \tan\theta = \frac{\sqrt{21}}{2}}$$

$\therefore \tan\left(\cos^{-1}\left(2/5\right)\right)$

$= \tan(\theta)$

$= \frac{x}{5} = \sqrt{21}/2$.

Example Evaluate $\cos(\tan^{-1}(x))$.

Method 1. Use trig identities.

Let $\theta = \tan^{-1}(x)$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\tan(\theta) = x$ (Find $\cos\theta$)

$$\sec^2\theta = 1 + \tan^2\theta$$
$$= 1 + x^2$$

$\sec\theta = \sqrt{1+x^2}$, since $\sec\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\cos\theta = \frac{1}{\sqrt{1+x^2}}$$

$$\cos(\tan^{-1}(x)) = \cos(\theta) = \frac{1}{\sqrt{1+x^2}}.$$

Method. graph.

$\cos(\tan^{-1}(x))$.

let $\theta = \tan^{-1}(x)$

$\tan(\theta) = x$

$$\cos(\theta) = \frac{1}{\sqrt{1+x^2}}$$