# §7.5 Strategies for Integration.

① Memorize the Table.

Last lecture we evaluated:

(i) $$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.$$

(ii) $$\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C.$$

NOTE: We showed:

$$\int \sec\theta\,d\theta = \frac{1}{2}\ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right| + C.$$

NOTE: using trig identities can show:

$$= \frac{1}{2}\ln\left|\frac{\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}-\frac{1}{\cos\theta}}\right| + C.$$

$$= \frac{1}{2} \ln \left| \frac{\tan\theta + \sec\theta}{\tan\theta - \sec\theta} \right| + C.$$

$$= \frac{1}{2} \ln \left| \frac{(\tan\theta + \sec\theta)(\tan\theta + \sec\theta)}{(\tan\theta - \sec(\theta))(\tan\theta + \sec\theta)} \right| + C$$

$$= \ln \left| \frac{(\tan\theta + \sec\theta)^2}{\underbrace{\cancel{\tan^2\theta - \sec^2\theta}}_{=-1}} \right|^{\frac{1}{2}} + C \quad (\text{But } |-1| = 1).$$

$$= \ln |\tan\theta + \sec\theta| + C = \int \sec\theta \, d\theta$$

(Both expressions are correct).

Back to Strategies.

② Simplify the Integrand (if possible)

e.g. $$\int \frac{x-2}{\sqrt[3]{x}} \, dx = \int \frac{x}{x^{1/3}} - \frac{2}{x^{1/3}} \, dx$$

$(\sqrt[3]{x} = x^{1/3})$. $$= \int x^{2/3} \, dx - 2\int x^{-1/3} \, dx.$$

Often, there is more than one way.

Example. $\int \frac{x}{x^2-1}\,dx$.

Method I. $\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$

$$A(x+1) + B(x-1) = x$$

$x=1$ : $2A = 1 \Rightarrow A = 1/2$.

$x=-1$ : $-2B = -1 \Rightarrow B = 1/2$.

$$\int \frac{x}{x^2-1}\,dx = \int \frac{1/2}{x-1}\,dx + \int \frac{1/2}{x+1}\,dx$$

$$= 1/2 \ln|x-1| + \frac{1}{2}\ln|x+1| + C.$$

Method 2 $\int \frac{x}{x^2-1}\,dx$ Subst $u = x^2-1$

$du = 2x\,dx$

$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln u + c.$$

$$= \frac{1}{2}\ln|x^2-1| + c$$

$$= \frac{1}{2}\left(\ln|(x-1)(x+1)|\right)$$

$$= \frac{1}{2}\left(\ln|x-1| + \ln|x+1|\right) + C$$

## ③ Obvious Substitution

$$\int f(x)\,\underbrace{g'(x)\,dx}_{du}$$ : Try $u = g(x)$

so that $du = g'(x)\,dx$

and this makes the integrand easier to integrate after the substitution.

④. Classify Integrand.

ⓐ Trig functions (see §7.2)

Know trig identities.

eg. $\Rightarrow \sin^2\theta + \cos^2\theta = 1.$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\sec^2\theta - 1 = \tan^2\theta$$

$$\tan^2\theta - \sec^2\theta = -1.$$

ⓑ Rational functions. (see §7.4)

partial fraction decomposition.

Know how to integrate $\dfrac{A}{x \pm a}$; $\dfrac{A}{(x\pm c)^n}$; $\dfrac{Ax+b}{(ax^2+bx+c)}$; $\dfrac{Ax+b}{(ax^2+bx+c)^2}$;

Know: Complete the square.

: Polynomial Division

## ③ Integration by Parts §7.1.

If $\int f(x)\,dx$ and $f(x)$ is a product of a power of $x$ or a polynomial.

and $e^x$

or trig.

or $\ln(x)$.

eg. $\int x^2 e^x\,dx$ → use integration by parts twice.

Choose. $\int f(x)\,g'(x)\,dx$

$f(x)$ so differentiation makes it simpler or not more complicated.

and $g(x)$ can be integrated.

$u = f(x)$ $\qquad$ $v = g(x)$

$du = f'(x)\,dx$ $\qquad$ $dv = g'(x)\,dx$

$$\int \underbrace{f(x)}_{u}\, g'\underbrace{(x)\,dx}_{dv} =$$

$$\boxed{\int u\,dv = uv - \int v\,du\,.}$$

Sometimes $\int f(x)\,dx = \int f(x)\cdot 1\,dx$

e.g. $\underline{f(x) = \ln x.}$

$u = f(x)$ $\qquad$ $v = x$

$du = f'(x)\,dx$ $\qquad$ $dv = 1\,dx$

$\int 1\,dx = x$

Might have to repeat integration by parts.

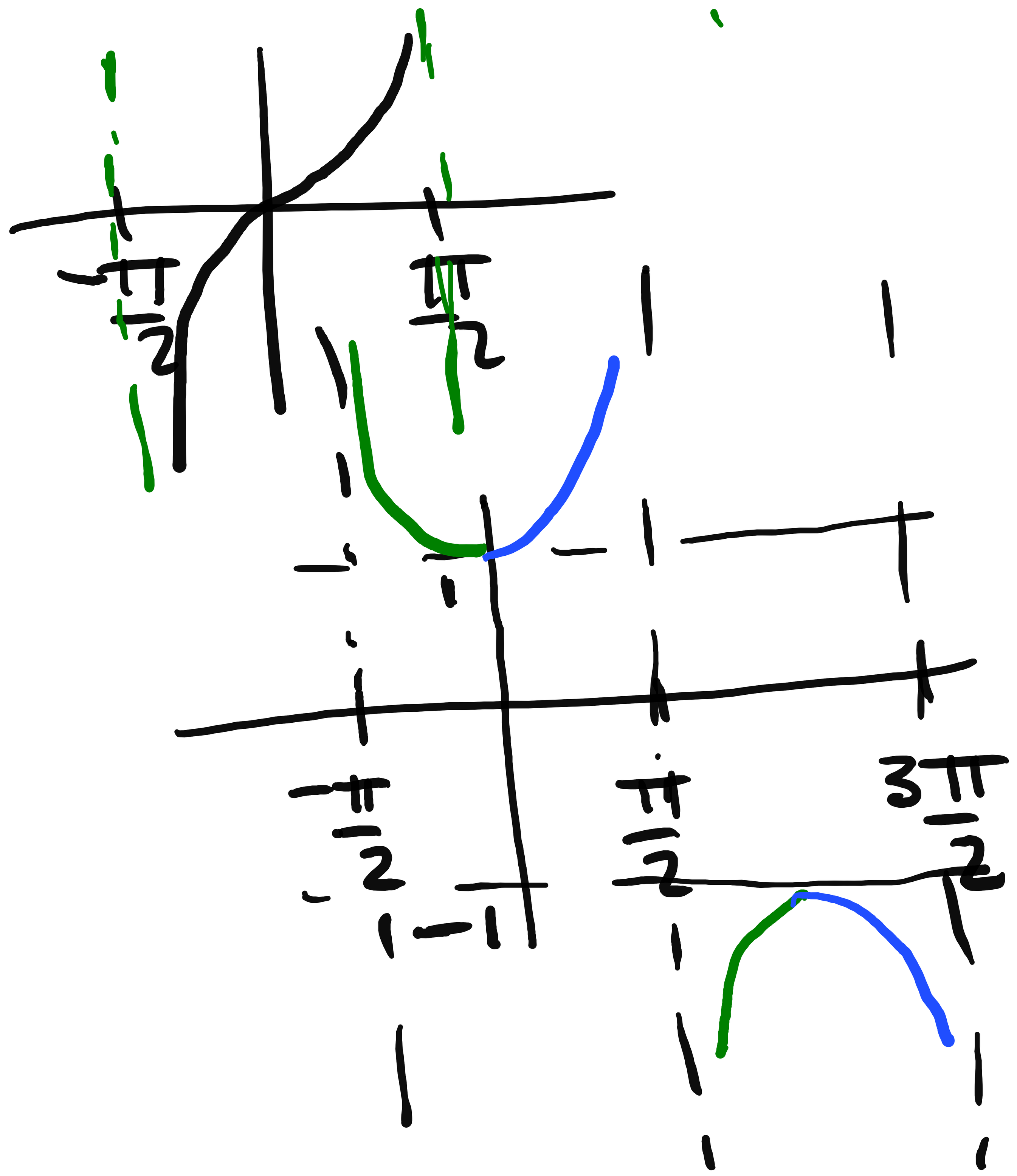
(d) Square root $\rightarrow \sqrt{\phantom{x}}$

trig subst.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta.$$
$$\sec^2\theta - 1 = \tan^2\theta.$$

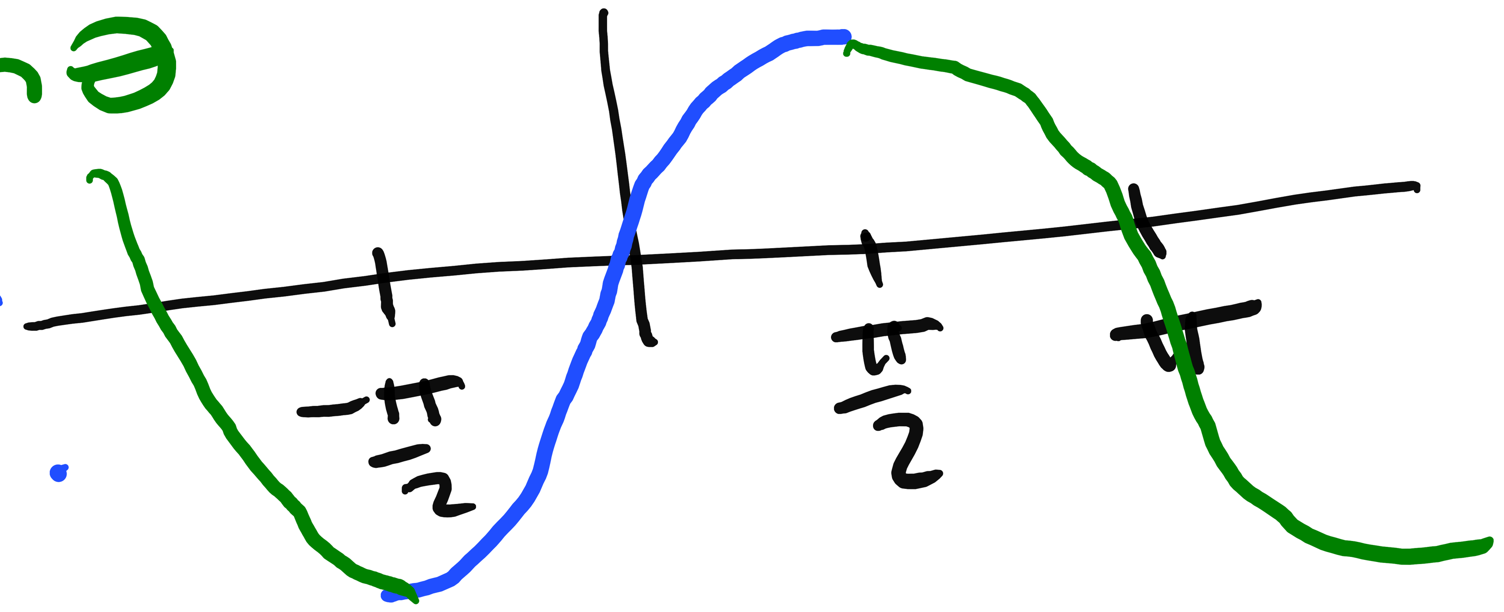
If

$\sqrt{x^2+a^2}$ $\quad x = a\tan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$




$\sqrt{x^2-a^2}$ $\quad x = a\sec\theta \quad 0 < \theta < \frac{\pi}{2}$ or $\pi < \theta < \frac{3\pi}{2}$

$\sqrt{a^2-x^2}$ $\quad x = a\sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$




Remember to go back to original variables if it is an indefinite integral.

If it is a definite integral, change limits of integration (or go back to original variables).

NOT responsible for RATIONALIZING SUBSTITUTION.

BEWARE: Not all integrands have anti-derivatives that are elementary functions.

Elementary functions are

- polynomials
- rational functions $\left( P(x)/Q(x) \right)$.
- $x^a$ (powers + roots)
- $a^x$
- $\log_a x$
- trig fcns.
- inverse trig functions.
- hyperbolic functions
- inverse hyperbolic functions

or functions obtained from these functions by $+, -, \times, \div,$ composition

Can alway differentiate such functions.

BUT $\int f(x)\,dx$ where $f(x)$ is an elementary function need NOT be an elementary function.

Examples of integrals that do not have anti derivatives that are elementary functions are numerous and includes:

$$\int e^{-x^2}\,dx, \quad \int \sin(x^2)\,dx, \quad \int \frac{1}{\sqrt{x^3+1}}\,dx,$$

$$\int x\tan x\,dx, \quad \int \frac{e^x}{x}\,dx.$$

$$\int \cos(e^x)\,dx.$$

$$\int \sqrt{x^3+1}\,dx.$$

$$\int \frac{1}{\ln(x)}\,dx. \quad \int \frac{\sin(x)}{x}\,dx$$

and many more.